Math 177: Homework N3

Due on Monday, May 28

1. A particle of mass $m$ is moving in $\mathbb{R}^3$ in a central field with potential energy $U(r)$. Write its Hamiltonian function and the equation of motion in the canonical coordinates $(r, \phi, \theta, p_r, p_\phi, p_\theta)$ associated with the spherical coordinates $(r, \phi, \theta)$.

2. The Lagrangian of a mechanical system is given by the formula

$$L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = (1 + q_1^2 + q_2^2) \left( \dot{q}_1^2 + 4 \dot{q}_1 \dot{q}_2 + 3 \dot{q}_2^2 \right) - (q_1 - q_2)^2 - (q_2 - q_3)^2 - (q_3 - q_1)^2.$$ 

Find the Hamiltonian function of the system.

3. Suppose that $\mathbb{R}^2$ is endowed with an area form $\omega = dp \wedge dq$. Let $H_t : \mathbb{R}^2 \to \mathbb{R}$, $t \in [0, 1]$, be a family of smooth functions equal to 0 outside of the unit disc $D$. Let $X_t := X_{H_t}$ be the Hamiltonian vector field generated by $H_t$, i.e. $X_t \omega = -dH_t$. Let $f_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the flow of area preserving transformations generated by $X_t$, i.e.

$$\frac{df_t}{dt}(x) = X_t(f_t(x)).$$

Let $z_0 \in \text{Int}D$ be a fixed point of $f_1$, i.e. $f_1(z_0) = z_0$. Denote by $\gamma$ the loop $\gamma : [0, 1] \to \mathbb{R}^2$ defined by the formula $\gamma(t) = f_t(z_0)$, $t \in [0, 1]$. Then the integral $S(z_0) := \int \gamma \, pdq - H_t dt$ is called action of the fixed point $z_0$.

Prove that for any path $\delta : [0, 1] \to \mathbb{R}^2$ such $\delta(0) \in \mathbb{R}^2 \setminus D$ and $\delta(1) = z_0$ one has

$$\int_{\delta} pdq - \int_{f_1(\delta)} pdq = S(z_0).$$
In particular, the integral in the left hand side of the equation is independent of the choice of the path \( \delta \), so that the action depends only on \( f_1 \) and not on a choice of the Hamiltonian \( H_t \) which generates it.

4. Prove the following Hamiltonian form of the least action principle. Consider a system given by a Hamiltonian function \( H(q, p) \) on the phase space \( T^* M \). Fix two points \( a, b \in M \) and denote by \( \mathcal{P} \) the space of all paths \( \gamma : [0, 1] \to T^* M \) with end points \( \gamma(0) \in T^*_a(M), \gamma(1) \in T^*_b(M) \). Prove that the trajectory of the system which starts at a point of \( T^*_a(M) \) and ends at a point of \( T^*_b(M) \) is an extremal of the action functions

\[
S(\gamma) = \int_{\gamma} pdq - H dt,
\]

where \( \gamma \in \mathcal{P} \).

5. Find an area preserving transformation \( f : \mathbb{R}^2 \to \mathbb{R}^2, (P, Q) = f(p, q) \), if its graph is given by the generating function \( F(q, P) = (q + q^3)P \). In other words, the graph of the area preserving map \( f \) in \( (\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2, dp \wedge dq - dP \wedge dQ) \) given by the generating function \( F \) with respect to the polarization of \( \mathbb{R}^4 \) by the coordinate plane \( (q, P) \) and \( (p, Q) \).

Each problem is 10 points.