Math 177: Homework N3

Due on Monday, May 28

1. A particle of mass m is moving in $\mathbb{R}^3$ in a central field with potential energy $U(r)$. Write its Hamiltonian function and the equation of motion in the canonical coordinates $(r, \phi, \theta, p_r, p_\phi, p_\theta)$ associated with the spherical coordinates coordinates (r, ϕ, θ).

2. The Lagrangian of a mechanical system is given by the formula

$$L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = (1 + q_1^2 + q_2^2)\left(\dot{q}_1^2 + 4\dot{q}_1\dot{q}_2 + 3\dot{q}_2^2\right) - (q_1 - q_2)^2 - (q_2 - q_3)^2 - (q_3 - q_1)^2.$$

Find the Hamiltonian function of the system.

3. Suppose that $\mathbb{R}^2$ is endowed with an area form $\omega = dp \wedge dq$. Let $H_t : \mathbb{R}^2 \to \mathbb{R}$, $t \in [0, 1]$, be a family of smooth functions equal to 0 outside of the unit disc D. Let $X_t := X_{H_t}$ be the Hamiltonian vector field generated by H_t, i.e. $X_t \lrcorner\, \omega = -dH_t$. Let $f_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the flow of area preserving transformations generated by X_t, i.e.

$$\frac{df_t}{dt}(x) = X_t(f_t(x)).$$

Let $z_0 \in \mathrm{Int}D$ be a fixed point of f_1, i.e. $f_1(z_0) = z_0$. Denote by γ the loop $\gamma : [0, 1] \to \mathbb{R}^2$ defined by the formula $\gamma(t) = f_t(z_0),\ t \in [0, 1]$. Then the integral $S(z_0) := \int\limits_{\gamma} pdq - H_t dt$ is called *action* of the fixed point z_0.

Prove that for any path $\delta : [0, 1] \to \mathbb{R}^2$ such $\delta(0) \in \mathbb{R}^2 \setminus D$ and $\delta(1) = z_0$ one has

$$\int\limits_{\delta} pdq - \int\limits_{f_1(\delta)} pdq = S(z_0).$$

In particular, the integral in the left hand side of the equation is independent of the choice of the path δ, so that the action depends only on f_1 and not on a choice of the Hamiltonian H_t which generates it.

4. Prove the following Hamiltonian form of the *least action principle*. Consider a system given by a Hamiltonian function $H(q,p)$ on the phase space T^*M. Fix two points $a, b \in M$ and denote by $\mathcal{P}$ the space of all paths $\gamma : [0,1] \to T^*M$ with end points $\gamma(0) \in T_a^*(M), \gamma(1) \in T_b^*(M)$. Prove that the trajectory of the system which starts at a point of $T_a^*(M)$ and ends at a point of $T_b^*(M)$ is an extremal of the action functions

$$S(\gamma) = \int_\gamma pdq - Hdt,$$

where $\gamma \in \mathcal{P}$.

5. Find an area preserving transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(P, Q) = f(p, q)$, if its graph is given by the generating function $F(q, P) = (q + q^3)P$. In other words, the graph of the area preserving map f in $(\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2, dp \wedge dq - dP \wedge dQ)$ given by the generating function F with respect to the polarization of $\mathbb{R}^4$ by the coordinate plane (q, P) and (p, Q).

Each problem is 10 points.