

Math 177: Homework N3

Solutions

1. A particle of mass m is moving in \mathbb{R}^3 in a central field with potential energy $U(r)$. Write its Hamiltonian function and the equation of motion in the canonical coordinates $(r, \phi, \theta, p_r, p_\phi, p_\theta)$ associated with the spherical coordinates (r, ϕ, θ) .

In the Cartesian coordinates the Lagrangian is equal

$$L = \frac{1}{2}\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - U(r),$$

where $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$ and hence the Hamiltonian is equal to

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + U(r). \quad (1)$$

We have

$$q_1 = r \sin \theta \cos \phi,$$

$$q_2 = r \sin \theta \sin \phi,$$

$$q_3 = r \cos \theta.$$

Thus

$$dq_1 = \sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi,$$

$$dq_2 = \sin \theta \sin \phi dr + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi,$$

$$dq_3 = \cos \theta dr - r \sin \theta d\theta.$$

Therefore

$$\begin{aligned}
p_1 dq_1 + p_2 dq_2 + p_3 dq_3 &= p_1(\sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi) \\
&+ p_2(\sin \theta \sin \phi dr + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi) + p_3(\cos \theta dr - r \sin \theta d\theta) \\
&= (p_1 \sin \theta \cos \phi + p_2 \sin \theta \sin \phi + p_3 \cos \theta) dr \\
&+ (p_1 r \cos \theta \cos \phi + p_2 r \cos \theta \sin \phi - p_3 r \sin \theta) d\theta \\
&+ (-p_1 r \sin \theta \sin \phi + p_2 r \sin \theta \cos \phi) d\phi.
\end{aligned}$$

We want in new coordinates $(r, \phi, \theta, p_r, p_\phi, p_\theta)$ to satisfy

$$p_1 dq_1 + p_2 dq_2 + p_3 dq_3 = p_r dr + p_\phi d\phi + p_\theta d\theta$$

, and hence we get

$$\begin{array}{lll}
p_1 \sin \theta \cos \phi & + p_2 \sin \theta \sin \phi + p_3 \cos \theta & = p_r \\
p_1 r \cos \theta \cos \phi & + p_2 r \cos \theta \sin \phi - p_3 r \sin \theta & = p_\theta \\
p_1(-r \sin \theta \sin \phi) & + p_2 r \sin \theta \cos \phi & = p_\phi.
\end{array}$$

Solving this linear system with respect to p_1, p_2 and p_3 and plugging the results into (1) we get

$$H = \frac{1}{2} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + U(r).$$

Alternatively one can first express \dot{q}_i in spherical coordinates, and then perform the Legendre transform.

2. The Lagrangian of a mechanical system is given by the formula

$$L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = (1 + q_1^2 + q_2^2) (\dot{q}_1^2 + 4\dot{q}_1 \dot{q}_2 + 3\dot{q}_2^2 + \dot{q}_3^2) - (q_1 - q_2)^2 - (q_2 - q_3)^2 - (q_3 - q_1)^2.$$

Find the Hamiltonian function of the system.

The Hamiltonian $H(q, p)$ is related to the Lagrangian $L(q, \dot{q})$ via the Legendre transform

$$H(q, p) = p\dot{q} - L(q, \dot{q}), \text{ where } p = \frac{\partial L}{\partial \dot{q}}.$$

In our case

$$\begin{aligned} p_1 &= (1 + q_1^2 + q_2^2)(2\dot{q}_1 + 4\dot{q}_2), \\ p_2 &= (1 + q_1^2 + q_2^2)(4\dot{q}_1 + 6\dot{q}_2) \\ p_3 &= 2(1 + q_1^2 + q_2^2)\dot{q}_3. \end{aligned}$$

Or,

$$\begin{aligned} \frac{p_1}{2(1 + q_1^2 + q_2^2)} &= \dot{q}_1 + 2\dot{q}_2, \\ \frac{p_2}{2(1 + q_1^2 + q_2^2)} &= 2\dot{q}_1 + 3\dot{q}_2 \\ \frac{p_3}{2(1 + q_1^2 + q_2^2)} &= \dot{q}_3. \end{aligned}$$

Solving the system with respect to \dot{q} we get

$$\begin{aligned} \dot{q}_1 &= \frac{-3p_1 + 2p_2}{2(1 + q_1^2 + q_2^2)}, \\ \dot{q}_2 &= \frac{2p_1 - p_2}{2(1 + q_1^2 + q_2^2)}, \\ \dot{q}_3 &= \frac{1}{2(1 + q_1^2 + q_2^2)}. \end{aligned}$$

Therefore,

$$\begin{aligned} H(q, p) &= \frac{1}{4(1 + q_1^2 + q_2^2)} \left(1 + (-3p_1 + 2p_2)^2 + 4(-3p_1 + 2p_2)(2p_1 - p_2) + 3(2p_1 - p_2)^2 + p_3^2 \right) \\ &+ (q_1 - q_2)^2 - (q_2 - q_3)^2 + (q_3 - q_1)^2 = \frac{1}{4(1 + q_1^2 + q_2^2)} \left(1 - 3p_1^2 - p_2^2 + 4p_1p_2 \right) \\ &+ (q_1 - q_2)^2 - (q_2 - q_3)^2 + (q_3 - q_1)^2. \end{aligned}$$

3. Suppose that \mathbb{R}^2 is endowed with an area form $\omega = dp \wedge dq$. Let $H_t : \mathbb{R}^2 \rightarrow \mathbb{R}$, $t \in [0, 1]$, be a family of smooth functions equal to 0 outside of the unit disc D . Let $X_t := X_{H_t}$ be the Hamiltonian vector field generated by H_t , i.e. $X_t \lrcorner \omega = -dH_t$. Let $f_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the flow of area preserving transformations generated by X_t , i.e.

$$\frac{df_t}{dt}(x) = X_t(f_t(x)).$$

Let $z_0 \in \text{Int}D$ be a fixed point of f_1 , i.e. $f_1(z_0) = z_0$. Denote by γ the loop $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ defined by the formula $\gamma(t) = f_t(z_0)$, $t \in [0, 1]$. Then the integral $S(z_0) := \int_{\gamma} pdq - \int_0^1 H_t dt$ is called *action* of the fixed point z_0 .

Prove that for any path $\delta : [0, 1] \rightarrow \mathbb{R}^2$ such $\delta(0) \in \mathbb{R}^2 \setminus D$ and $\delta(1) = z_0$ one has

$$-\int_{\delta} pdq + \int_{f_1(\delta)} pdq = S(z_0).$$

In particular, the integral in the left hand side of the equation is independent of the choice of the path δ , so that the action depends only on f_1 and not on a choice of the Hamiltonian H_t which generates it.

Denote $Q := \{0 \leq s, t \leq 1\} \subset \mathbb{R}^2$ and consider a map $F : Q \times \mathbb{R}^2$ given by the formula

$$Q(s, t) = f_t(\delta(s)).$$

Consider the form $\lambda := F^*pdq$ and apply to its Stokes' theorem

$$\int_{\partial Q} \lambda = \int_Q d\lambda.$$

Note that

$$\int_{\partial Q} \lambda = \int_{\delta} pdq + \int_{\gamma} pdq - \int_{f_1(\gamma)} pdq. \quad (2)$$

On the other hand,

$$\int_Q d\lambda = \int_Q dF^*pdq = \int_Q F^*dp \wedge dq = \int_0^1 \left(\int_0^1 \begin{vmatrix} \frac{\partial p}{\partial s} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{vmatrix} ds \right) dt \quad (3)$$

But

$$\frac{\partial p}{\partial t} = \dot{p} = -\frac{\partial H}{\partial q}, \frac{\partial q}{\partial t} = \dot{q} = \frac{\partial H}{\partial p}.$$

Therefore,

$$\begin{aligned}
\int_Q F^* dp \wedge dq &= \int_0^1 \left(\int_0^1 \left| \begin{array}{cc} \frac{\partial p}{\partial s} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{array} \right| ds \right) dt = \int_0^1 \left(\int_0^1 \left| \begin{array}{cc} \frac{\partial p}{\partial s} & -\frac{\partial H}{\partial q} \\ \frac{\partial q}{\partial s} & \frac{\partial H}{\partial p} \end{array} \right| ds \right) dt \\
&= \int_0^1 \left(\int_0^1 \left(\frac{\partial H}{\partial p} \frac{\partial p}{\partial s} + \frac{\partial H}{\partial q} \frac{\partial q}{\partial s} \right) ds \right) dt = \int_0^1 \left(\int_0^1 \frac{\partial H(p(s,t), q(s,t))}{\partial s} ds \right) dt \\
&= \int_0^1 (H(p(1,t), q(1,t)) - H(p(0,t), q(0,t))) dt.
\end{aligned} \tag{4}$$

We have $(p(0,t), q(0,t)) = f_t(\delta(0)) = \delta(0)$, and

$$(p(1,t), q(1,t)) = f_t(z_0) = \gamma(t).$$

Hence

$$\int_0^1 (H(p(1,t), q(1,t)) - H(p(0,t), q(0,t))) dt = \int_0^1 H(\gamma(t)) dt. \tag{5}$$

Combining equations (2)-(5)

$$-\int_{\delta} pdq + \int_{f_1(\gamma)} pdq = \int_{\gamma} pdq - H dt.$$

4. Prove the following Hamiltonian form of the *least action principle*. Consider a system given by a Hamiltonian function $H(q, p)$ on the phase space T^*M . Fix two points $a, b \in M$ and denote by \mathcal{P} the space of all paths $\gamma : [0, 1] \rightarrow T^*M$ with end points $\gamma(0) \in T_a^*(M)$, $\gamma(1) \in T_b^*(M)$. Prove that the trajectory of the system which starts at a point of $T_a^*(M)$ and ends at a point of $T_b^*(M)$ is an extremal of the action functions

$$S(\gamma) = \int_{\gamma} pdq - H dt,$$

where $\gamma \in \mathcal{P}$.

Let us deduce the Euler Lagrange equations for the above variational problem for the Hamiltonian action functional $S(\gamma) = \int p dq - H dt$. Take a variation $\gamma_\epsilon(t) := \gamma(t) + \epsilon h(t)$, where $h(t) = (p = h_1(t), q = h_2(t))$ and $h_2(0) = h_2(1) = 0$. Then

$$\begin{aligned} S(\gamma_\epsilon) - S(\gamma) &= \int_0^1 \left((p(t) + \epsilon h_1(t))(\dot{q}(t) + \epsilon \dot{h}_2(t)) - p(t)\dot{q}(t) \right) dt \\ &+ \int_0^1 (H(p(t) + \epsilon h_1(t), q(t) + \epsilon h_2(t)) - H(p(t), q(t))) dt \\ &= \epsilon \left(\int_0^1 (p\dot{h}_2 + \dot{q}h_1) dt - \int_0^1 \left(\frac{\partial H}{\partial p} h_1 + \frac{\partial H}{\partial q} h_2 \right) dt \right) + o(\epsilon). \end{aligned}$$

Integration by part and taking into account the boundary conditions $h_2(0) = h_2(1) = 0$ we get

$$\int_0^1 p\dot{h}_2 dt = - \int_0^1 \dot{p}h_2 dt.$$

Therefore,

$$\begin{aligned} S(\gamma_\epsilon) - S(\gamma) &= \\ &= \epsilon \left(\int_0^1 (-\dot{p}h_2 + \dot{q}h_1) dt - \int_0^1 \left(\frac{\partial H}{\partial p} h_1 + \frac{\partial H}{\partial q} h_2 \right) dt \right) + o(\epsilon) \\ &= \epsilon \left(\int_0^1 \left(-\dot{p} - \frac{\partial H}{\partial q} \right) h_1 + \left(\dot{q} - \frac{\partial H}{\partial p} \right) h_2 dt \right) + o(\epsilon). \end{aligned}$$

The differential (or it is called in calculus of variations, the *variation* δS , i.e. the linear part of the action functional is equal to

$$\delta S = \int_0^1 \left(-\dot{p} - \frac{\partial H}{\partial q} \right) h_1 + \left(\dot{q} - \frac{\partial H}{\partial p} \right) h_2 dt.$$

At critical points $\delta S = 0$ for any h_1, h_2 , and hence this leads to the equations

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}.$$

But this is just the Hamiltonian system of equations, and hence the variational principle is equivalent to Hamiltonian equations.

5. Find an area preserving transformation $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $(P, Q) = f(p, q)$, if its graph is given by the generating function $F(q, P) = (q + q^3)P$. In other words, the graph of the area preserving map f in $(\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2, dp \wedge dq - dP \wedge dQ)$ given by the generating function F with respect to the polarization of \mathbb{R}^4 by the coordinate plane (q, P) and (p, Q) .

We have

$$\begin{aligned} p &= \frac{\partial F}{\partial q} = P(1 + 3q^2), \\ Q &= \frac{\partial F}{\partial P} = q + q^3. \end{aligned}$$

Solving with respect to variables P, Q we get

$$\begin{aligned} P &= \frac{p}{1 + 3q^2}, \\ Q &= q + q^3. \end{aligned}$$

These formulas define the required canonical transformation.