Math 177: Homework N2

Due on Wednesday, May 14

1. Prove the following properties of the Lie bracket of two vector fields:

a) *Jacobi identity:*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

b) Given a function f and vector fields X, Y one has

$$[X, fY] = df(X)Y + f[X,Y].$$

2. Let α be a non-vanishing differential 1-form and X a vector field on $\mathbb{R}^2$ Show that in order that the flow X^t of X was a symmetry of the line field $\ell = \{\alpha = 0\}$ it is necessary and sufficient that $L_X \alpha = f\alpha$ for some function f on $\mathbb{R}^2$.

3. Find a curve on the plane whose tangent lines form with the coordinate axis triangles of area $2a^2$.

4. Let us view $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ with the operator of multiplication by i. Denote by

- $GL(2n)$ the group of invertible real linear transformations of $\mathbb{R}^{2n}$;
- $GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$ the group of invertible complex linear transformation of $\mathbb{C}^n$;
- $U(n) \subset GL(n,\mathbb{C})$ the group of unitary transformations of $\mathbb{C}^n$, i.e. transformations preserving the Hermitian form $\sum\limits_1^n z_k \overline{w_k}$;

- $Sp(2n) \subset GL(2n, \mathbb{R})$ the group of symplectic transformations of $\mathbb{R}^{2n}$, i.e. transformations preserving the symplectic form $\omega = \sum_1^n x_k \wedge y_k$;
- $SO(2n) \subset GL(2n, \mathbb{R})$ the group of orthogonal transformations of $\mathbb{R}^{2n}$, i.e. transformations which preserves the standard scalar dot-product.

Prove that

$$SO(2n) \cap Sp(2n) = GL(n, \mathbb{C}) \cap SO(2n) = GL(n, \mathbb{C}) \cap Sp(2n) = U(n).$$

5. Consider a PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u - xy.$$

Using the method of characteristics solve the Cauchy problem for the initial data $u(2, y) = 1 + y^2$.

6. Find the largest value of t for which the solution of the Cauchy problem for the partial differential equation

$$u_t + uu_x = -\sin x, \quad u|_{t=0} = 0,$$

can be extended to the interval $[0, t)$.

Each problem is 10 points