Math 177: Homework N1

Solutions

1. Consider a system of two quasi-homogeneous equations

$$xy\frac{dy}{dx} = z^2,$$
$$z\frac{dz}{dx} = xy, \quad x, y, z > 0.$$

Find a change of variables which reduces it to a first order system.

Rewrite the system in the Pfaffian form

$$xydy = z^2dx,$$
$$zdz = xydx.$$

First determine the weights Assign $x \to \alpha, y \to \beta, z \to \gamma$. Then

$$2\beta = 2\gamma, 2\gamma = 2\alpha + \beta,$$

i.e. we can choose $\alpha = 1, \beta = \gamma = 2$. Choose new coordinates $u = x, v = \frac{y}{x^2}, s = \frac{z}{x^2}$. Then $y = u^2v, z = u^2s$, and hence the system takes the form

$$u^3v(2uvdu + u^2dv) - u^4s^2du = 0,$$
$$u^2s(2sudu + u^2ds) - u^3vdu = 0,$$

or simplifying the equations we get

$$(2v^2 - s^2)du + uvdv = 0,$$
$$(2s^2 - v)du + suds = 0.$$

Hence,

$$\frac{du}{u} = \frac{v}{s^2 - 2v^2}dv =$$
$$\frac{du}{u} = \frac{s}{v - 2s^2}ds.$$

Hence, we reduced the system to 1 first order equation

$$\frac{v}{s^2 - 2v^2}dv = \frac{s}{v - 2s^2}ds.$$

After solving we find v, s, and then we can find the remaining variable u from the equation

$$\frac{du}{u} = \frac{s}{v - 2s^2}ds.$$

2. A particle is moving in a central potential field $V(x) = C||x||^\alpha$ in $\mathbb{R}^3$. Suppose that there exists a homothety $x \mapsto \lambda x$, $x \in \mathbb{R}^3$, which maps a periodic orbit Γ_1 onto a periodic orbit Γ_2. Suppose that the periods of the orbits are T_1 and T_2, respectively. Find α.

The Newton equation of motion is $\ddot{x} = -\nabla V(x) = C\alpha||x||^{\alpha-1}e_r,$, where e_r is the radial unit vector field. Reducing to the first order system we get

$$\dot{x} = y,$$
$$\dot{y} = C\alpha||x||^{\alpha-1}e_r$$

, or

$$dx = ydt,$$
$$dy = C\alpha||x||^{\alpha-1}e_r dt$$

Assigning weights $x \to a$, $y \to b$, $t \to c$, we get:

$$a = b + c, b = (\alpha - 1)a + c,$$

i.e. $a(2-\alpha) = 2c$. We can take $a = 1, c = \frac{2-\alpha}{2}$. That means that the $x \mapsto \lambda x, t \mapsto \lambda^{\frac{2-\alpha}{2}}$ is a symmetry of the system. This implies that $T_2 = \lambda^{\frac{2-\alpha}{2}} T_1$, and hence $\frac{2-\alpha}{2} = \log_\lambda \frac{T_2}{T_1} = \frac{\ln T_2 - \ln T_1}{\ln \lambda}$, or

$$\alpha = \frac{2\ln\lambda}{\ln T_1 - \ln T_2} - 2.$$

3. Compute the phase flow of the vector field on the plane

$$v = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}.$$

Suppose that a line field

$$\ell := \{a(x,y)dx + b(x,y)dy = 0\}$$

is invariant under this flow. Which change of variables one needs to make in order to solve the equation

$$a(x,y)dx + b(x,y)dy = 0.$$

The phase flow of v given by solving the system

$$\dot{x} = y,$$

$$\dot{y} = x$$

Solving it we get

$$x(t) = x_0 \cosh t + y_0 \sinh t,$$

$$y(t) = x_0 \sinh t + y_0 \cosh t,$$

The trajectories of this system is the families of hyperbolas

$$x^2 - y^2 = C.$$

. It is impossible to find a curve transverse to all trajectories. So the change of variables has to be done separately in quadrants We will do it the quadrant $|x| < y$.

Then the ray $r_0 := \{y = 0, x \geq 0\}$ is transverse to all the trajectories. After the time t flow the ray r_0 is mapped onto the ray $r_t = \{y = (\tanh t)x\}$. We note that Hence, as the new coordinatex in which variable separate we can choose $u = yx, v = x^2 = y^2$.

4. Find the solutions, the criminant and the discriminant of the equation

$$(y')^3 + 3xy' - 3y = 0.$$

Consider $\Sigma = \{p^3 + 3xp - 3y = 0\} \subset J^1(\mathbb{R})$. The characteristics are given by the system of equations

$$F(x, y, p) = p^3 + 3xp - 3y = 0,$$
$$dF = 3(p^2 + x)dp + 3pdx - 3dy = 0.$$
$$dy - pdx = 0.$$

Hence,

$$(p^2 + x)dp = 0.$$

Hence, there are solutions for $dp = 0$. They are lines

$$y = Cx + \frac{C^3}{3}.$$

There also the solution $x = -p^2$. This equation together with the equation of Σ defines the criminant. The discriminant is its projection to the (x, y)-plane, which is given by

$$y^2 = \frac{4}{9}x^3.$$

5. Let $f = \sum_1^n a_{ij}x_ix_j$ be a quadratic form on $\mathbb{R}^n$. Show that its Legendre transform $g(p)$ is again a quadratic form $g(p) = \sum_1^n b_{ij}p_ip_j$ and that the value of these forms at the points corresponding to each other under the Legendre map coincide.

(Legendre map is a diffeomorphism $x \mapsto p$ which is used in the definition of Legendre transform.)

We can assume that A is a symmetric matrix (because one can always rewrite a quadratic form with a symmetric matrix). We will also assume that A is not degenerate because otherwise the Legendre transform is not well defined. Then

$$f(x) = \frac{1}{2}\langle Ax, x\rangle.$$

The factor $\frac{1}{2}$ is inserted to simplify computations. Legendre transform $\hat{f}(p)$ is given by

$$\hat{f}(p) = \langle x, p\rangle - \frac{1}{2}\langle Ax, x\rangle,$$

where $p = \nabla f(x) = Ax$,, or $x = A^{-1}p$.. Hence,

$$\hat{f}(p) = \langle A^{-1}p, p\rangle - \frac{1}{2}\langle p, A^{-1}p\rangle = \frac{1}{2}\langle A^{-1}p, p\rangle.$$

Under the Legendre map $p = Ax$ we have

$$\hat{f}(p) = \hat{f}(Ax) = \frac{1}{2}\langle x, Ax\rangle = f(x).$$

Each problem is 10 points.