Math 177. Geometric Methods in ODE

Spring 2018

Take-home Final Exam

Due on Monday, June 4

1. Prove that the following Cauchy problem :

$$(x^3 - 3xy^2)\frac{\partial u}{\partial x} + (3x^2y - y^3)\frac{\partial u}{\partial y} = 0, \quad u|_{x^2+y^2=2} = \sin y,$$

has no solution in a neighborhood of the point $(1,1)$.

2. Consider a Lagrangian system in the upper half plane

$$\{(x,y);\ y > 0\} \subset \mathbb{R}^2$$

with the Lagrangian function

$$L(x,y,\dot{x},\dot{y}) = \frac{\dot{x}^2 + \dot{y}^2}{y^2}.$$

Write the equation of motion in the Hamiltonian form, find explicitly all the trajectories and describe qualitatively their behavior.

3. Consider the contact plane field $\xi = \{dz - y dx = 0\}$ in $\mathbb{R}^3$. A 1 dimensional submanifold $\Gamma \subset \mathbb{R}^3$ is called Legendrian if it is tangent to ξ. Denote by π the projection $(x,y,z) \mapsto (x,y)$. Suppose that the submanifold Γ is connected and closed (i.e. diffeomorphic to a circle). Prove that the projected curve $\pi(\Gamma) \subset \mathbb{R}^2$ must have self intersection points.

Each problem is 10 points.