1. Prove that the following Cauchy problem:

\[
(x^3 - 3xy^2) \frac{\partial u}{\partial x} + (3x^2y - y^3) \frac{\partial u}{\partial y} = 0, \quad u|_{x^2+y^2=2} = \sin y,
\]

has no solution in a neighborhood of the point (1, 1).

At the point \(x = 1, y = 1\) the equation reads

\[
-2 \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0,
\]

or

\[
du(v) = 0, \quad \text{where} \ v = (-1, 1).
\]

But the vector \(v\) is tangent to the circle \(\{x^2 + y^2 = 2\}\) at the point (1, 1), and hence

\[
\frac{du(v)}{dy} \bigg|_{y=1} = -\cos 1 \neq 0.
\]

2. Consider a Lagrangian system in the upper half plane

\[
\{(x, y); \ y > 0\} \subset \mathbb{R}^2
\]

with the Lagrangian function

\[
L(x, y, \dot{x}, \dot{y}) = \frac{\dot{x}^2 + \dot{y}^2}{y^2}.
\]

Write the equation of motion in the Hamiltonian form, find explicitly all the trajectories and describe qualitatively their behavior.

We have

\[
H = \frac{y^2(p_x^2 + p_y^2)}{4}.
\]
The corresponding Hamiltonian system is

\[
\dot{x} = \frac{y^2 p_x}{2} \\
\dot{y} = \frac{y^2 p_y}{2} \\
\dot{p}_y = -\frac{y(p_x^2 + p_y^2)}{2} \\
\dot{p}_x = 0
\]

Hence we have 2 integrals, \( H \) and \( p_x \). Recall that \( p_x = \frac{\partial L}{\partial x} = \frac{\dot{x} y^2}{y^2} \). Hence, if \( p_x \neq 0 \) then

\[
\frac{4H}{p_x^2} = \frac{y^2 (\dot{x}^2 + \dot{y}^2)}{\dot{x}^2} = \text{const.}
\]

The following picture illustrates the meaning of the quantity \( \frac{y^2 (\dot{x}^2 + \dot{y}^2)}{\dot{x}^2} \).

Thus, \( \frac{4H}{p_x^2} \) is the distance from the point \((x, y)\) in the configuration space to the point of intersection of the \(x\)-axis with the normal to the trajectory \((x(t), y(t))\). In other words, all trajectories with \( p_x \neq 0 \) are semicircles with their center at a point of the \(x\)-axis.
If $p_x = 0$ then from the Hamiltonian system (1) we get
\[
\begin{align*}
\dot{x} &= 0 \\
\dot{y} &= \frac{y^2 p_y}{2} \\
\dot{p}_y &= -\frac{y (p_y^2)}{2} \\
\dot{p}_x &= 0.
\end{align*}
\]
From the integral $H = E$ we get $p_y = \frac{2\sqrt{E}}{y}$, and hence the second equation (1) takes the form
\[
\dot{y} = \sqrt{Ey}.
\]
Hence, $x = \text{const}$, $y = Ce^{\sqrt{Et}}$, and therefore the trajectory is the ray orthogonal to the $x$-axis.

3. Consider the contact plane field $\xi = \{dz - ydx = 0\}$ in $\mathbb{R}^3$. A 1 dimensional submanifold $\Gamma \subset \mathbb{R}^3$ is called Legendrian if it is tangent to $\xi$. Denote by $\pi$ the projection $(x, y, z) \mapsto (x, y)$. Suppose that the submanifold $\Gamma$ is connected and closed (i.e. diffeomorphic to a circle). Prove that the projected curve $\pi(\Gamma) \subset \mathbb{R}^2$ must have self intersection points.

Denote the projection of the curve to the $(x, y)$-plane by $\Gamma$. Note that
\[
\int_{\Gamma} (dz - ydx) = 0,
\]
because $\Gamma$ is tangent to $\xi = \{dz - ydx = 0\}$.
But
\[
\int_{\Gamma} (dz - ydx) = \int_{\Gamma} dz - \int_{\Gamma} ydx.
\]
The first integral is equal to 0, because the integral of an exact 1-form over a closed curve vanishes. On the other hand,
\[
\int_{\Gamma} ydx = \int_{\Gamma} ydx.
\]
But the integral in the right-hand-side is the area enclosed by the curve $\Gamma$, and hence if $\Gamma$ had no self-intersection points this would imply that $\int_{\Gamma} ydx \neq 0$. 