Math 177. Geometric Methods in ODE

Spring 2018

Take-home Final Exam

1. Prove that the following Cauchy problem :

$$(x^3 - 3xy^2)\frac{\partial u}{\partial x} + (3x^2 y - y^3)\frac{\partial u}{\partial y} = 0, \quad u|_{x^2+y^2=2} = \sin y,$$

has no solution in a neighborhood of the point $(1, 1)$.

At the point $x = 1, y = 1$ the equation reads

$$-2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0,$$

or

$$du(v) = 0, \quad \text{where} v = (-1, 1).$$

But the vector v is tangent to the circle $\{x^2 + y^2 = 2\}$ at the point $(1, 1)$, and hence

$$du(v) = \frac{d(\sin y)}{dy}\Big|_{y=1} = -\cos 1 \neq 0.$$

2. Consider a Lagrangian system in the upper half plane

$$\{(x, y);\ y > 0\} \subset \mathbb{R}^2$$

with the Lagrangian function

$$L(x, y, \dot{x}, \dot{y}) = \frac{\dot{x}^2 + \dot{y}^2}{y^2}.$$

Write the equation of motion in the Hamiltonian form, find explicitly all the trajectories and describe qualitatively their behavior.

We have

$$H = \frac{y^2(p_x^2 + p_y^2)}{4}.$$

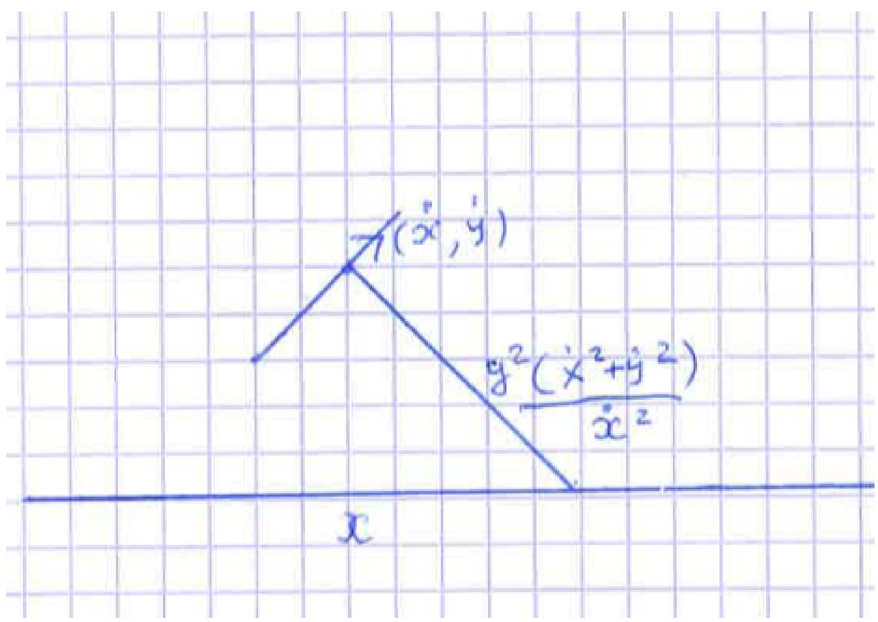
The corresponding Hamiltonian system is

$$\dot{x} = \frac{y^2 p_x}{2} \tag{1}$$
$$\dot{y} = \frac{y^2 p_y}{2} \tag{2}$$
$$\dot{p}_y = -\frac{y(p_x^2 + p_y^2)}{2} \tag{3}$$
$$\dot{p}_x = 0. \tag{4}$$

Hence we have 2 integrals, H and p_x. Recall that $p_x = \frac{\partial L}{\partial x} = \frac{2\dot{x}}{y^2}$. Hence, if $p_x \neq 0$ then

$$\frac{4H}{p_x^2} = \frac{y^2(\dot{x}^2 + \dot{y}^2)}{\dot{x}^2} = \text{const.}$$

The following picture illustrates the meaning of the quantity $\frac{y^2(\dot{x}^2+\dot{y}^2)}{\dot{x}^2}$.



Thus, $\frac{4H}{p_x^2}$ is the distance from the point (x, y) in the configuration space to the point of intersection of the x-axis with the normal to the trajectory $(x(t), y(t))$. In other words, all trajectories with $p_x \neq 0$ are semicircles with their center at a point of the x-axis.

If $p_x = 0$ then from the Hamiltonian system (1) we get

$$\dot{x} = 0$$
$$\dot{y} = \frac{y^2 p_y}{2}$$
$$\dot{p}_y = -\frac{y(p_y^2)}{2}$$
$$\dot{p}_x = 0.$$

From the integral $H = E$ we get $p_y = \frac{2\sqrt{E}}{y}$, and hence the second equation (1) takes the form

$$\dot{y} = \sqrt{E}y.$$

Hence, $x = \text{const}$, $y = Ce^{\sqrt{E}t}$, and therefore the trajectory is the ray orthogonal to the x-axis.

3. Consider the contact plane field $\xi = \{dz - ydx = 0\}$ in $\mathbb{R}^3$. A 1 dimensional submanifold $\Gamma \subset \mathbb{R}^3$ is called Legendrian if it is tangent to ξ. Denote by π the projection $(x, y, z) \mapsto (x, y)$. Suppose that the submanifold Γ is connected and closed (i.e. diffeomorphic to a circle). Prove that the projected curve $\pi(\Gamma) \subset \mathbb{R}^2$ must have self intersection points.

Denote the projection of the curve to the (x, y)-plane by $\overline{\Gamma}$. Note that

$$\int\limits_{\Gamma} (dz - ydx) = 0,$$

because Γ is tangent to $\xi = \{dz - ydx = 0\}$.
But

$$\int\limits_{\Gamma} (dz - ydx) = \int\limits_{\Gamma} dz - \int\limits_{\Gamma} ydx.$$

The first integral is equal to 0, because the integral of an exact 1-form over a closed curve vanishes. On the other hand,

$$\int\limits_{\Gamma} ydx = \int\limits_{\overline{\Gamma}} ydx.$$

But the integral in the right-hand-side is the area enclosed by the curve $\overline{\Gamma}$, and hence if $\overline{\Gamma}$ had no self-intersection points this would imply that $\int\limits_{\overline{\Gamma}} ydx \neq 0$.