Symplectic Geometry of Stein Manifolds

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Incomplete draft, October 12, 2007

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Chapter 1

J-convex functions and hypersurfaces

1.1 Linear algebra

A *complex vector space* (V, J) is a real vector space V of dimension $2n$ with an endomorphism J satisfying $J^2 = -\mathbb{1}$. A *Hermitian form* on (V, J) is an $\mathbb{R}$-bilinear map $H : V \times V \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first variable and satisfies $H(X, Y) = \overline{H(Y, X)}$. If H is, moreover, positive definite it is called *Hermitian metric*. We can write a Hermitian form H uniquely as

$$H = g - i\omega,$$

where g is a symmetric and ω a skew-symmetric bilinear form on the real vector space V. The forms g and ω determine each other:

$$g(X, Y) = \omega(X, JY), \qquad \omega(X, Y) = g(JX, Y)$$

for $X, Y \in V$. Moreover, the forms ω and g are invariant under J, which can be equivalently expressed by the equation

$$\omega(JX, Y) + \omega(X, JY) = 0.$$

Conversely, given a skew-symmetric J-invariant form ω, we can uniquely reconstruct the corresponding Hermitian form H:

$$H(X, Y) := \omega(X, JY) - i\omega(X, Y). \tag{1.1}$$

For example, consider the complex vector space $(\mathbb{C}^n, i)$ with coordinates $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$. It carries the standard Hermitian metric

$$(v, w) := \sum_{j=1}^{n} v_j \bar{w}_j = \langle v, w \rangle - i\omega_0(v, w),$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean metric and $\omega_0 = \sum_j dx_j \wedge dy_j$ the standard symplectic form on $\mathbb{C}^n$.

1.2 J-convex functions

An *almost complex structure* on a smooth manifold V of real dimension $2n$ is an automorphism $J : TV \to TV$ satisfying $J^2 = -\mathbb{1}$ on each fiber. The pair (V, J) is called *almost complex manifold*. It is called *complex manifold* if J is *integrable*, i.e. J is induced by complex coordinates on V. By the theorem of Newlander and Nirenberg [38], a (sufficiently smooth) almost complex structure J is integrable if and only if its *Nijenhuis tensor*

$$N(X,Y) := [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y], \qquad X, Y \in TV,$$

vanishes identically.

In the following let (V, J) be an almost complex manifold. To a smooth function $\phi : V \to \mathbb{R}$ we associate the 2-form

$$\omega_\phi := -dd^{\mathbb{C}}\phi,$$

where the differential operator $d^{\mathbb{C}}$ is defined by

$$d^{\mathbb{C}}\phi(X) := d\phi(JX)$$

for $X \in TV$. The form ω_ϕ is in general not J-invariant. However, it is J-invariant if J is integrable. To see this, consider the complex vector space $(\mathbb{C}^n, i)$. Given a function $\phi : \mathbb{C}^n \to \mathbb{R}$, define the complex valued $(1,1)$-form

$$\partial\bar\partial\phi := \sum_{i,j=1}^{n} \frac{\partial^2\phi}{\partial z_i \partial \bar z_j} dz_i \wedge d\bar z_j.$$

Using the identities

$$dz_j \circ i = i\, dz_j, \qquad d\bar z_j \circ i = -i\, d\bar z_j$$

we compute

$$d^{\mathbb{C}}\phi = \sum_j \frac{\partial\phi}{\partial z_j} dz_j \circ i + \frac{\partial\phi}{\partial \bar z_j} d\bar z_j \circ i = \sum_j i\frac{\partial\phi}{\partial z_j} dz_j - i\frac{\partial\phi}{\partial \bar z_j} d\bar z_j,$$

$$dd^{\mathbb{C}}\phi = -2i \sum_{i,j} \frac{\partial^2\phi}{\partial z_i \partial \bar z_j} dz_i \wedge d\bar z_j.$$

Hence

$$\omega_\phi = 2i\partial\bar\partial\phi \tag{1.2}$$

and the i-invariance of ω_ϕ follows from the invariance of $\partial\bar\partial\phi$.

A function ϕ is called *J-convex*[1] if $\omega_\phi(X, JX) > 0$ for all nonzero tangent vectors X. If ω_ϕ is J-invariant it defines by (1.1) a unique Hermitian form

$$H_\phi := g_\phi - i\omega_\phi,$$

and ϕ is J-convex iff the Hermitian form H_ϕ is positive definite.

From (1.2) we can derive a simple expression for the form H_ϕ associated to a function $\phi : \mathbb{C}^n \to \mathbb{R}$ in terms of the matrix $a_{ij} := \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}$. For $v, w \in \mathbb{C}^n$ we have

$$\begin{aligned}\omega_\phi(v, w) &= 2i \sum_{ij} a_{ij} dz_i \wedge d\bar{z}_j(v, w) = 2i \sum_{ij} a_{ij}(v_i \bar{w}_j - w_i \bar{v}_j) \\ &= 2i \sum_{ij} (a_{ij} v_i \bar{w}_j - \bar{a}_{ij} \bar{v}_i w_j) = -4\mathrm{Im}\left(\sum_{ij} a_{ij} v_i \bar{w}_j\right),\end{aligned}$$

hence

$$H_\phi(v, w) = 4 \sum_{i,j=1}^{n} \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j} v_i \bar{w}_j. \tag{1.3}$$

Example 1.1. The function $\phi(z) := \sum_j |z_j|^2$ on $\mathbb{C}^n$ is i-convex with respect to the standard complex structure i. The corresponding form H_ϕ equals $4(\, , \,)$, where $(\, , \,)$ is the standard Hermitian metric on $\mathbb{C}^n$.

1.3 The Levi form of a hypersurface

Let Σ be a smooth (real) hypersurface in an almost complex manifold (V, J). Each tangent space $T_p\Sigma \subset T_pV$, $p \in \Sigma$, contains a unique maximal complex subspace $\xi_p \subset T_p\Sigma$ which is given by

$$\xi_p = T_p\Sigma \cap JT_p\Sigma.$$

Suppose that Σ is cooriented by a transverse vector field ν to Σ in V such that $J\nu$ is tangent to Σ. The hyperplane field ξ can be defined by a Pfaffian equation $\{\alpha = 0\}$, where the sign of the 1-form α is fixed by the condition $\alpha(J\nu) > 0$. The 2-form

$$\omega_\Sigma := d\alpha|_\xi$$

is then defined uniquely up to multiplication by a positive function. As in the previous section we may ask whether ω_Σ is J-invariant. The following lemma gives a necessary and sufficient condition in terms of the Nijenhuis tensor.

[1] Throughout this book, by convexity and J-convexity we will always mean *strict* convexity and J-convexity. Non-strict (J-)convexity will be referred to as *weak* (J-)convexity.

Lemma 1.2. *The form ω_Σ is J-invariant for a hypersurface Σ if and only if $N|_{\xi\times\xi}$ takes values in ξ. The form ω_Σ is J-invariant for* every *hypersurface Σ if and only if for all $X, Y \in TV$, $N(X,Y)$ lies in the complex plane spanned by X and Y. In particular, this is the case if J is integrable or if V has complex dimension 2.*

Proof. Let $\Sigma \subset V$ be a hypersurface and α a defining 1-form for ξ. Extend α to a neighborhood of Σ such that $\alpha(\nu) = 0$. For $X, Y \in \xi$ we have $[X, Y] \in T\Sigma$ and therefore $J[X, Y] = a\nu + Z$ for some $a \in \mathbb{R}$ and $Z \in \xi$. This shows that

$$\alpha(J[X, Y]) = 0$$

for all $X, Y \in \xi$. Applying this to various combinations of X, Y, JX and JY we obtain

$$\alpha\Big(N(X, Y)\Big) = \alpha([JX, JY]) - \alpha([X, Y]),$$
$$\alpha\Big(JN(X, Y)\Big) = \alpha([X, JY]) + \alpha([JX, Y]).$$

The form ω_Σ is given by

$$\omega_\Sigma(X, Y) = \frac{1}{2}\Big(X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X, Y]\Big) = -\frac{1}{2}\alpha([X, Y]).$$

Inserting this in the formulae above yields

$$-\frac{1}{2}\alpha\Big(N(X, Y)\Big) = \omega_\Sigma(JX, JY) - \omega_\Sigma(X, Y),$$
$$-\frac{1}{2}\alpha\Big(JN(X, Y)\Big) = \omega_\Sigma(X, JY) + \omega_\Sigma(JX, Y).$$

Hence the J-invariance of ω_Σ is equivalent to

$$\alpha\Big(N(X, Y)\Big) = \alpha\Big(JN(X, Y)\Big) = 0,$$

i.e. $N(X, Y) \in \xi$ for all $X, Y \in \xi$. This proves the first statement and the 'if' in the second statement. For the 'only if' it suffices to note that if $N(X, Y)$ does not lie in the complex plane spanned by X and Y for some $X, Y \in TV$, then we find a hypersurface Σ such that $X, Y \in \xi$ and $N(X, Y) \notin \xi$. □

A hypersurface Σ is called *Levi-flat* if $\omega_\Sigma \equiv 0$. It is called *J-convex* (or *strictly pseudoconvex*) if $\omega_\Sigma(X, JX) > 0$ for all nonzero $X \in \xi$. If ω_Σ is J-invariant it defines a Hermitian form L_Σ on ξ by the formula

$$L_\Sigma(X, Y) := \omega_\Sigma(X, JY) - i\omega_\Sigma(X, Y)$$

for $X, Y \in \xi$. The Hermitian form L_Σ is called the *Levi form* of the (cooriented) hypersurface Σ. As pointed out above, it is defined uniquely up to multiplication by a positive function. Note that Σ is Levi-flat iff $L_\Sigma \equiv 0$, and J-convex iff L_Σ is positive definite. We will sometimes also refer to ω_Σ as the Levi form.

If the hypersurface Σ is given by an equation $\{\phi = 0\}$ for a function $\phi : V \to \mathbb{R}$, then we can choose $\alpha = -d^{\mathbb{C}}\phi$ as the 1-form defining ξ (with the coorientation of Σ given by $d\phi$). Thus the Levi form can be defined as

$$\omega_\Sigma(X, Y) = -dd^{\mathbb{C}}\phi(X, Y).$$

This shows that regular level sets of a J-convex function ϕ are J-convex (being cooriented by $d\phi$). It turns out that the converse is also almost true (similarly to the situation for convex functions and hypersurfaces).

Lemma 1.3. *Let $\phi : V \to \mathbb{R}$ be a smooth function on an almost complex manifold without critical points such that all its level sets are compact and J-convex. Then there exists a convex increasing function $f : \mathbb{R} \to \mathbb{R}$ such that the composition $f \circ \phi$ is J-convex.*

Proof. Consider a regular level set Σ of ϕ. For a function $f : \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned} d^{\mathbb{C}}(f \circ \phi) &= f' \circ \phi \, d^{\mathbb{C}}\phi, \\ -dd^{\mathbb{C}}(f \circ \phi) &= -f'' \circ \phi \, d\phi \wedge d^{\mathbb{C}}\phi - f' \circ \phi \, dd^{\mathbb{C}}\phi. \end{aligned}$$

By the J-convexity of Σ, the term $-f' \circ \phi \; dd^{\mathbb{C}}\phi$ is positive definite on the maximal complex subspace $\xi \subset \Sigma$ if $f' > 0$. Let ν be the normal vector to Σ with $d\phi(\nu) = 1$. Then $-d\phi \wedge d^{\mathbb{C}}\phi(\nu, J\nu) = 1$, and by compactness of the level sets,

$$-dd^{\mathbb{C}}(f \circ \phi)(\nu, J\nu) > f'' \circ \phi - h \circ \phi \, f' \circ \phi$$

for some smooth function $h : \mathbb{R} \to (0, \infty)$. Now solve the differential equation $f''(y) = h(y)f'(y)$ with initial condition $f'(y_0) > 0$. The solution exists for all $y \in \mathbb{R}$ and satisfies $f' > 0$, so $f \circ \phi$ is J-convex. □

Remark 1.4. The proof of the preceding lemma also shows: If $\phi : V \to \mathbb{R}$ is J-convex, then $f \circ \phi$ is J-convex for any function $f : \mathbb{R} \to \mathbb{R}$ with $f' > 0$ and $f'' \geq 0$.

A vector field is called *complete* if its flow exists for all forward and backward times. For a J-convex function ϕ, let $\nabla_\phi \phi$ be the gradient of ϕ with respect to the metric $g_\phi = \omega_\phi(\cdot, J\cdot)$. In general, $\nabla_\phi \phi$ need not be complete:

Example 1.5. The function $\phi(z) := \sqrt{1 + |z|^2}$ on $\mathbb{C}$ satisfies

$$\frac{\partial^2 \phi}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z} \frac{z}{\sqrt{1 + |z|^2}} = \frac{1}{\sqrt{1 + |z|^2}^3},$$

so $g_\phi = 4(1 + |z|^2)^{-3/2}\langle \, , \, \rangle$, where $\langle \, , \, \rangle$ is the standard metric. In particular, ϕ is i-convex. Its gradient is determined from

$$d\phi = \frac{x\,dx + y\,dy}{\sqrt{1 + |z|^2}} = \frac{4}{\sqrt{1 + |z|^2}^3}\langle \nabla_\phi \phi, \cdot \rangle,$$

thus $\nabla_\phi\phi = \frac{1+|z|^2}{4}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$. A gradient line $\gamma(t)$ with $|\gamma(0)| = 1$ is given by $\gamma(t) = h(t)\gamma(0)$, where $h(t)$ satisfies $h' = \frac{1+h^2}{4}h$. This shows that $\gamma(t)$ tends to infinity in finite time, hence the gradient field $\nabla_\phi\phi$ is not complete.

However, the gradient field $\nabla_\phi\phi$ can always be made complete by composing ϕ with a sufficiently convex function:

Proposition 1.6. *Let $\phi : V \to [a,\infty)$ be an exhausting J-convex function on an almost complex manifold. Then for any diffeomorphism $f : [a,\infty) \to [b,\infty)$ such that $f'' > 0$ and $\lim_{y\to\infty} f'(y) = \infty$, the function $f \circ \phi$ is J-convex and its gradient vector field is complete.*

Proof. The function $\psi := f \circ \phi$ satisfies

$$dd^{\mathbb{C}}\psi = f'' \circ \phi \; d\phi \wedge d^{\mathbb{C}}\phi + f' \circ \phi \; dd^{C}\phi\,.$$

In particular, ψ is J-convex if $f' > 0$ and $f'' > 0$. The metric associated to ψ is given by

$$\begin{aligned} g_\psi(X,Y) &= -dd^{\mathbb{C}}\psi(X,JY) \\ &= +f'' \circ \phi \; [d\phi(X)d\phi(Y) + d^{\mathbb{C}}\phi(X)d^{C}\phi(Y)] + f' \circ \phi \; g_\phi(X,Y)\,. \end{aligned}$$

Let us compute the gradient $\nabla_\psi\psi$. We will find it in the form

$$\nabla_\psi\psi = \lambda\nabla_\phi\phi$$

for a function $\lambda : V \to \mathbb{R}$. The gradient is determined by

$$g_\psi(\nabla_\psi\psi, Y) = d\psi(Y) = f' \circ \phi \; d\phi(Y)$$

for any vector $Y \in TV$. Using $d\phi(\nabla_\phi\phi) = g_\phi(\nabla_\phi\phi, \nabla_\phi\phi) =: |\nabla_\phi\phi|^2$ and $d^{\mathbb{C}}\phi(\nabla_\phi\phi) = g_\phi(\nabla_\phi\phi, J\nabla_\phi\phi) = 0$, we compute the left hand side as

$$\begin{aligned} &g_\psi(\nabla_\psi\psi, Y) \\ &= \lambda\Big\{f'' \circ \phi \; [d\phi(\nabla_\phi\phi)d\phi(Y) + d^{\mathbb{C}}\phi(\nabla_\phi\phi)d^{C}\phi(Y)] + f' \circ \phi \; g_\phi(\nabla_\phi\phi, Y)\Big\} \\ &= \lambda\{f'' \circ \phi \; |\nabla_\phi\phi|^2 d\phi(Y) + f' \circ \phi \; d\phi(Y)\}. \end{aligned}$$

Comparing with the right side, we find

$$\lambda = \frac{f' \circ \phi}{f'' \circ \phi \; |\nabla_\phi\phi|^2 + f' \circ \phi}.$$

Since ϕ is proper, we only need to check completeness of the gradient flow for positive times. Consider an unbounded gradient trajectory $\gamma : [0,T) \to V$, i.e., a solution of

$$\frac{d\gamma}{dt}(t) = \nabla_\phi\phi\big(\gamma(t)\big), \qquad \lim_{t\to T} \phi\big(\gamma(t)\big) = \infty.$$

Here T can be finite or $+\infty$. The function ϕ maps the image of γ diffeomorphically onto some interval $[c,\infty)$. It pushes forward the vector field $\nabla_\phi\phi$ (which is tangent to the image of γ) to the vector field

$$\phi_*(\nabla_\phi\phi) = h(y)\frac{\partial}{\partial y},$$

where t and y are the coordinates on $[0,T)$ and $[c,\infty)$, respectively, and

$$h(y) := |\nabla_\phi\phi|^2\left(\phi^{-1}(y)\right) > 0.$$

Similarly, ϕ pushes forward $\nabla_\psi\psi = \lambda\nabla_\phi\phi$ to the vector field

$$\phi_*(\nabla_\psi\psi) = \lambda\left(\phi^{-1}(y)\right)h(y)\frac{\partial}{\partial y} = \frac{f'(y)h(y)}{f''(y)h(y)+f'(y)}\frac{\partial}{\partial y} =: v(y).$$

Hence completeness of the vector field $\nabla_\psi\psi$ on the trajectory γ is equivalent to the completeness of the vector field v on $[c,\infty)$. An integral curve of v satisfies $\frac{dy}{ds} = v(y)$, or equivalently,

$$ds = \frac{f''(y)h(y)+f'(y)}{f'(y)h(y)}dy.$$

Thus completeness of the vector field v is equivalent to

$$+\infty = \int_c^\infty \frac{f''(y)h(y)+f'(y)}{f'(y)h(y)}dy = \int_c^\infty \frac{f''(y)dy}{f'(y)} + \int_c^\infty \frac{dy}{h(y)}.$$

The first integral on the right hand side is equal to $\int_c^\infty d\big(\ln f'(y)\big)$, so it diverges if and only if $\lim_{y\to\infty} f'(y) = \infty$. □

We will call an exhausting J-convex function *completely exhausting* if its gradient vector field $\nabla_\phi\phi$ is complete.

1.4 J-convexity and geometric convexity

Next we investigate the relation between i-convexity and geometric convexity. Consider $\mathbb{C}^n = \mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{R}$ with coordinates $(z_1,\dots,z_{n-1},u+iv)$. Let $\Sigma\subset\mathbb{C}^n$ be a hypersurface which is given as a graph $\{u = f(z,v)\}$ for some smooth function $f:\mathbb{C}^{n-1}\times\mathbb{R}\to\mathbb{R}$. Assume that $f(0,0)=0$ and $df(0,0)=0$. Every hypersurface in a complex manifold can be locally written in this form.

The Taylor polynomial of second order of f around $(0,0)$ can be written as

$$T_2f(z,v) = \sum_{i,j} a_{ij}z_i\bar z_j + 2Re\sum_{i,j} b_{ij}z_iz_j + v\,l(z,\bar z) + cv^2, \tag{1.4}$$

where l is some linear function of z and $\bar{z}$, and $a_{ij} = \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(0,0)$. Let Σ be cooriented by the gradient of the function $f(z,v) - u$. Then the 2-form ω_Σ at the point 0 is given on $X, Y \in \xi_0 = \mathbb{C}^{n-1}$ by

$$\begin{aligned}\omega_\Sigma(X,Y) = 2i\partial\bar{\partial}f(X,Y) = 2i\sum_{i,j} a_{ij}dz_i \wedge d\bar{z}_j(X,Y) \\ = -4\mathrm{Im}\,(AX,Y),\end{aligned}$$

where A is the complex $(n-1)\times(n-1)$ matrix with entries a_{ij}. Hence the Levi form at 0 is

$$L_\Sigma = 4\langle A\cdot,\cdot\rangle.$$

If the function f is (strictly) convex, then

$$T_2 f(z,0) + T_2 f(iz,0) = 2\sum_{ij} a_{ij}z_i\bar{z}_j > 0$$

for all $z \neq 0$, so the Levi form is positive definite. This shows that convexity of Σ implies i-convexity. The converse is not true, see the examples below. It is true, however, locally after a biholomorphic change of coordinates.

Proposition 1.7 (R.Narasimhan). *A hypersurface $\Sigma \subset \mathbb{C}^n$ is i-convex if and only if it can be made (strictly) convex in a neighborhood of each of its points by a biholomorphic change of coordinates.*

Proof. The 'if' follows from the discussion above and the invariance of J-convexity under biholomorphic maps. For the converse write Σ in local coordinates as a graph $\{u = f(z,v)\}$ as above and consider its second Taylor polynomial (1.4). Let $w = u + iv$, and perform in a neighborhood of 0 the holomorphic change of coordinates $\tilde{w} := w - 2\sum_{ij} b_{ij}z_iz_j$. Then

$$\tilde{u} = \sum a_{ij}z_i\bar{z}_j + \tilde{v}\,l(z,\bar{z}) + c\tilde{v}^2 + O(3).$$

After another local change of coordinates $w' := \tilde{w} - \lambda\tilde{w}^2$, $\lambda \in \mathbb{R}$, we have

$$u' = \tilde{u} + \lambda(v')^2 + O(3) = \sum a_{ij}z_i\bar{z}_j + v'l(z,\bar{z}) + (c+\lambda)(v')^2 + O(3).$$

For λ sufficiently large the quadratic form on the right hand side is positive definite, so the hypersurface Σ is convex in the coordinates (z,w'). □

Consider for a moment a cooriented hypersurface in $\mathbb{R}^n$ with the Euclidean metric $\langle\ ,\ \rangle$. Its *second fundamental form*

$$II : T\Sigma \times T\Sigma \to \mathbb{R}$$

can be defined as follows. For $X \in T_x\Sigma$ let $\gamma : (-\epsilon,\epsilon) \to \Sigma$ be a curve with $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. Then

$$II(X,X) := -\langle\ddot{\gamma}(0),\nu\rangle,$$

where ν is the unit normal vector to Σ in x defining the coorientation. The matrix representing the second fundamental form equals the differential of the Gauss map which associates to every point its unit normal vector. Our sign convention is chosen in such a way that the unit sphere in $\mathbb{R}^n$ has positive principal curvatures if it is cooriented by the *outward* pointing normal vector field. The *mean curvature* along a k-dimensional subspace $S \subset T_x\Sigma$ is defined as

$$\frac{1}{k}\sum_{i=1}^{k} II(v_i, v_i)$$

for some orthonormal basis $v_1, \ldots, v_k$ of S. If Σ is given as a graph $\{x_n = f(x_1, \ldots, x_{n-1})\}$ with $f(0) = 0$ and $df(0) = 0$, then for $X \in \mathbb{R}^{n-1}$ we can choose the curve

$$\gamma(t) := \Big(tX, f(tX)\Big)$$

in Σ. Taking the second derivative we obtain

$$II(X, X) = \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) X_i X_j, \tag{1.5}$$

if Σ is cooriented by the gradient of the function $f - x_n$. This leads to the following geometric characterization of i-convexity:

Proposition 1.8. *The Levi form of a cooriented hypersurface $\Sigma \subset \mathbb{C}^n$ with respect to the standard complex structure i is given at a point $z \in \Sigma$ by*

$$L_\Sigma(X, X) = \frac{1}{2}\Big(II(X, X) + II(iX, iX)\Big)$$

for $X \in T_z\Sigma$. Thus Σ is i-convex if and only if at every point $z \in \Sigma$ the mean curvature along any complex line in $T_z\Sigma$ is positive.

Proof. Write Σ locally as a graph $\{u = f(z, v)\}$ with $f(0,0) = 0$ and $df(0,0) = 0$, and such that the gradient of $f - u$ defines the coorientation of Σ. Consider the second Taylor polynomial (1.4) of f in $(0,0)$. In view of (1.5), the mean curvature along the complex line generated by $X \in \mathbb{C}^{n-1}$ is given by

$$\begin{aligned} \frac{1}{2}\Big(II(X, X) + II(iX, iX)\Big) &= \frac{1}{2}\Big(T_2 f(X) + T_2 f(iX)\Big) \\ &= \sum_{ij} a_{ij} X_i \bar{X}_j = L_\Sigma(X, X), \end{aligned}$$

and the proposition follows. □

1.5 Examples of J-convex functions and hypersurfaces

Quadratic functions. For the function

$$\phi(z) := \sum_{k=1}^{n} \lambda_k x_k^2 + \mu_k y_k^2$$

on $\mathbb{C}^n$ we have

$$\omega_\phi = 2\sum_k (\lambda_k + \mu_k) dx_k \wedge dy_k.$$

So ϕ is J-convex if and only if

$$\lambda_k + \mu_k > 0 \text{ for all } k = 1, \dots, n. \tag{1.6}$$

Consider a level set Σ of ϕ in the case $\lambda_k > 0$ and $\mu_k < 0$. The intersection of Σ with any plane in the x-coordinates is a curve with positive curvature determined by the λ_k. The intersection with a plane in the y-coordinates has negative curvature determined by the y_k. The condition (1.6) assures that along any complex line these curvatures add up to a positive mean curvature.

Totally real submanifolds. A submanifold L of an almost complex manifold (V, J) is called *totally real* if it has no complex tangent lines, i.e. $J(TL) \cap TL = \{0\}$ at every point. This condition implies $\dim_\mathbb{R} L \le \dim_\mathbb{C} V$. For example, the linear subspaces $\mathbb{R}^k := \{(x_1, \dots, x_k, 0, \dots, 0) \mid x_i \in \mathbb{R}\} \subset \mathbb{C}^n$ are totally real for all $k = 0, \dots, n$. If we have an Hermitian metric on (V, J) we can define the distance function $dist_L : V \to \mathbb{R}$,

$$dist_L(x) := \inf\{dist(x, y) \mid y \in L\}.$$

Proposition 1.9. *Let L be a totally real submanifold of an almost complex manifold (V, J). Then the squared distance function $dist_L^2$ with respect to any Hermitian metric on V is J-convex in a neighborhood of L. In particular, if L is compact, then $\{\mathrm{dist}_L \le \varepsilon\}$ is a tubular neighbourhood of L with J-convex boundary for each sufficiently small $\varepsilon > 0$.*

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Proof. Let $Q : T_pV \to \mathbb{R}$ be the Hessian quadratic form of $dist_L^2$ at a point $p \in L$. Its value $Q(z)$ equals the squared distance of $z \in T_pV$ from the linear subspace $T_pL \subset T_pV$. Choose an orthonormal basis $e_1, Je_1, \dots, e_n, Je_n$ of T_pV such that $e_1, \dots, e_k$ is a basis of T_pL. In this basis,

$$Q\Big(\sum_{i=1}^{n}(x_i e_i + y_i Je_i)\Big) = \sum_{j>k} x_j^2 + \sum_{i=1}^{n} y_i^2,$$

which is J-convex by Example 1. So $dist_L^2$ is J-convex on L and therefore by continuity in a neighborhood of L. $\square$

Remark 1.10. (1) The last statement of Proposition 1.9 extends to the non-compact case: *Every properly embedded totally real submanifold of an almost complex manifold has an arbitrarily small tubular neighbourhoods with J-convex boundary.* Of course the radius of the neighborhood may go to 0 at infinity

(2) Proposition 1.9 can be generalized as follows. Let W be *any* compact submanifold of an almost complex manifold (V, J), and suppose that a function $\phi : V \to \mathbb{R}$ satisfies the following J-convexity condition on W: *The form $-dd^{\mathbb{C}}\phi$ is positive on any complex line tangent to W.* Note that this condition is vacuously satisfied for any function on a totally real manifold. Choose any Hermitian metric on V. Then the function $\phi + \lambda dist^2_W$ is J-convex in a neighborhood of W for a sufficiently large positive λ. If W is non-compact (but properly embedded), then we need to choose as λ not a constant but a positive function $\lambda : W \to \mathbb{R}$ which may grow at infinity.

Holomorphic line bundles. A complex line bundle $\pi : E \to V$ over a complex manifold V is called *holomorphic line bundle* if the total space E is a complex manifold and the bundle possesses holomorphic local trivializations. For a Hermitian metric on $E \to V$ consider the hypersurface

$$\Sigma := \{z \in E \,\Big|\, |z| = 1\} \subset E.$$

Complex multiplication $U(1) \times \Sigma \to \Sigma$, $(e^{i\theta}, z) \mapsto e^{i\theta} \cdot z$ provides Σ with the structure of a $U(1)$ principal bundle over V. Let α be the 1-form on Σ defined by

$$\alpha\Big(\frac{d}{d\theta}\Big|_0 e^{i\theta} \cdot z\Big) = 1, \qquad \alpha|_\xi = 0,$$

where ξ is the distribution of maximal complex subspaces of $T\Sigma$. The imaginary valued 1-form $i\alpha$ defines the unique connection on the $U(1)$ principal bundle $\Sigma \to V$ for which all horizontal subspaces are J-invariant. Its curvature is the imaginary valued (1,1)-form Ω on V satisfying $\pi^*\Omega = d(i\alpha)$. On the other hand, α is a defining 1-form for the hyperplane distribution $\xi \subset T\Sigma$, so $\omega_\Sigma = d\alpha|_{\xi\times\xi}$ defines the Levi form of Σ. Thus ω_Σ and the curvature form Ω are related by the equation

$$i\omega_\Sigma(X, Y) = \Omega(\pi_* X, \pi_* Y) \tag{1.7}$$

for $X, Y \in \xi$. The line bundle $E \to V$ is called *positive* (*resp. negative*) if it admits a Hermitian metric such that the corresponding curvature form Ω satisfies

$$\frac{i}{2\pi}\Omega(X, JX) > 0 \;\big(\text{resp. } < 0\big)$$

for all $0 \neq X \in TV$. Since π is holomorphic, equation (1.7) implies

Proposition 1.11. *Let $E \to V$ be a holomorphic line bundle over a complex manifold. There exists a Hermitian metric on $E \to V$ such that the hypersurface $\{z \in E \,\big|\, |z| = 1\}$ is J-convex if and only if E is a negative line bundle.*

If V is compact, then the closed 2-form $\frac{i}{2\pi}\Omega$ represents the first Chern class $c_1(E)$,

$$\left[\frac{i}{2\pi}\Omega\right] = c_1(E)$$

(see [32], Chapter 12). Conversely, for every closed (1,1)-form $\frac{i}{2\pi}\Omega$ representing $c_1(E)$, Ω is the curvature of some Hermitian connection $i\alpha$ as above ([21], Chapter 1, Section 2). So a line bundle over V is positive/negative if and only if its first Chern class can be represented by a positive/negative (1,1)-form. If V has complex dimension 1 we get a very simple criterion.

Corollary 1.12. *Let V be a compact Riemann surface and $[V] \in H_2(V, \mathbb{R})$ its fundamental class. A holomorphic line bundle $E \to V$ admits a Hermitian metric such that the hypersurface $\{z \in E \;\big|\; |z| = 1\}$ is J-convex if and only if $c_1(E) \cdot [V] < 0$.*

For example, the corollary applies to the tangent bundle of a Riemann surface of genus ≥ 2.

Proof. Since $H^2(V, \mathbb{R})$ is 1-dimensional, $c_1(E) \cdot [V] < 0$ if and only if $c_1(E)$ can be represented by a negatively oriented area form. But any negatively oriented area form on V is a negative (1,1)-form. □

Remark 1.13. If $E \to V$ is just a complex line bundle (i.e. not holomorphic), then the total space E does not carry a natural almost complex structure. Such a structure can be obtained by choosing a Hermitian connection on $E \to V$ and taking the horizontal spaces as complex subspaces with the complex multiplication induced from V via the projection. If we fix an almost complex structure on the total space E such that the projection π is J-holomorphic, then Proposition 1.9 remains valid.

1.6 J-convex functions and hypersurfaces in $\mathbb{C}^n$

Consider $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ with complex coordinates $(z_1, \dots, z_{n-1}, w)$. Let $\Sigma \subset \mathbb{C}^n$ be a hypersurface which is given by an equation

$$\Psi(z, w) = 0,$$

with $\Psi_w \neq 0$, and cooriented by the gradient vector $\nabla\Psi = 2(\Psi_{\bar{z}_1}, \dots, \Psi_{\bar{z}_{n-1}}, \Psi_{\bar{w}})$. We will derive a criterion for i-convexity of Σ in terms of Ψ. The maximal complex subspace in Σ is given by $\xi = \{v \in \mathbb{C}^n \mid (v, \nabla\Psi) = 0\}$, where $(\ ,\ )$ is the standard Hermitian product on $\mathbb{C}^n$. It is generated by the vectors

$$v_j := \Psi_w e_j - \Psi_{z_j} e_n, \qquad j = 1, \dots, n-1,$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{C}^n$. To $a \in \mathbb{C}^{n-1}$ associate the vector

$$b := \sum_{j=1}^{n-1} a_j v_j = (a_1 \Psi_w, \dots, a_{n-1}\Psi_w, -\sum_{j=1}^{n-1} a_j \Psi_{z_j}) \in \xi.$$

By (1.3), the Levi form of Σ is given on b by

$$\begin{aligned} \frac{1}{4} L_\Sigma(b,b) = \sum_{j=1}^{n} \Psi_{z_i \bar z_j} b_i \bar b_j &= |\Psi_w|^2 \sum_{i,j=1}^{n-1} \Psi_{z_i \bar z_j} a_i \bar a_j \\ &- 2\mathrm{Re} \sum_{i,j=1}^{n-1} \Psi_{z_i \bar w} \Psi_w \Psi_{\bar z_j} a_i \bar a_j + \Psi_{w\bar w} \sum_{i,j=1}^{n-1} \Psi_{z_i} \Psi_{\bar z_j} a_i \bar a_j. \end{aligned}$$

First consider the case $n = 2$. Denote complex coordinates on $\mathbb{C}^2$ by (ζ, w). By the preceding discussion, i-convexity of Σ amounts to the inequality

$$\begin{aligned} \mathcal{L}(\Psi) &:= 4L_\Sigma\Big((\Psi_w, -\Psi_\zeta), (\Psi_w, -\Psi_\zeta)\Big) \\ &= 16\Big(\Psi_{\zeta\bar\zeta}|\Psi_w|^2 - 2\mathrm{Re}\,(\Psi_{\zeta\bar w}\Psi_w\Psi_{\bar\zeta}) + \Psi_{w\bar w}|\Psi_\zeta|^2\Big) > 0. \end{aligned}$$

We will refer to $\mathcal{L}(\Psi)$ as the *Levi form* of the function Ψ. Now return to the case of general dimension n. For $a \in \mathbb{C}^{n-1}$ consider the function

$$\Psi^a(\zeta, w) := \Psi(\zeta a_1, \dots, \zeta a_{n-1}, w).$$

Its derivatives are given by

$$\Psi^a_\zeta = \sum_{i=1}^{n-1} \Psi_{z_i} a_i, \qquad \Psi^a_{\bar\zeta} = \sum_{i=1}^{n-1} \Psi_{\bar z_i} \bar a_i, \qquad \Psi^a_{\zeta\bar\zeta} = \sum_{i,j=1}^{n-1} \Psi_{z_i \bar z_j} a_i \bar a_j,$$

$$\Psi^a_w = \Psi_w, \qquad \Psi^a_{\bar w} = \Psi_{\bar w}, \qquad \Psi^a_{w\bar w} = \Psi_{w\bar w}, \qquad \Psi^a_{\zeta\bar w} = \sum_{i=1}^{n-1} \Psi_{z_i \bar w} a_i.$$

Inspection of the expression above shows that

$$\frac{1}{4} L_\Sigma(b,b) = \Psi^a_{\zeta\bar\zeta}|\Psi^a_w|^2 - 2\mathrm{Re}\,(\Psi^a_{\zeta\bar w}\Psi^a_w\Psi^a_{\bar\zeta}) + \Psi^a_{w\bar w}|\Psi^a_\zeta|^2 = \frac{1}{16}\mathcal{L}(\Psi^a).$$

So we have shown

Lemma 1.14. *A hypersurface $\{\Psi(z_1, \dots, z_{n-1}, w) = 0\}$ in $\mathbb{C}^n$ with $\Psi_w \neq 0$ is i-convex (cooriented by $\nabla\Phi$) if and only if for every unit vector $a \in \mathbb{C}^{n-1}$,*

$$\mathcal{L}(\Psi^a) := 16\Big(\Psi^a_{\zeta\bar\zeta}|\Psi^a_w|^2 - 2\mathrm{Re}\,(\Psi^a_{\zeta\bar w}\Psi^a_w\Psi^a_{\bar\zeta}) + \Psi^a_{w\bar w}|\Psi^a_\zeta|^2\Big) > 0.$$

Let us consider once more the case $n = 2$. We denote complex coordinates in $\mathbb{C}^2$ by $z = (\zeta, w)$ with $\zeta = s + it$, $w = u + iv$. Suppose that a hypersurface $\Sigma \subset \mathbb{C}^2$ is given as a graph

$$\Psi(\zeta, w) := \psi(\zeta, u) - v = 0.$$

Then

$$\begin{gathered}
2\Psi_{\bar\zeta} = \psi_s + i\psi_t, \qquad 4\Psi_{\zeta\bar\zeta} = \psi_{ss} + \psi_{tt}, \qquad 4|\Psi_\zeta|^2 = \psi_s^2 + \psi_t^2, \\
2\Psi_w = \psi_u + i, \qquad 4\Psi_{w\bar w} = \psi_{uu}, \qquad 4|\Psi_w|^2 = 1 + \psi_u^2, \\
4\Psi_{\zeta\bar w} = \psi_{su} - i\psi_{tu}, \qquad 4\Psi_w\Psi_{\bar\zeta} = (\psi_u\psi_s - \psi_t) + i(\psi_s + \psi_u\psi_t), \\
16 Re\Psi_{\zeta\bar w}\Psi_w\Psi_{\bar\zeta} = \psi_{su}(\psi_u\psi_s - \psi_t) + \psi_{tu}(\psi_s + \psi_u\psi_t),
\end{gathered}$$

thus we have proved

Lemma 1.15. *The Levi form of the hypersurface $\{v = \psi(s,t,u)\} \subset \mathbb{C}^2$, cooriented by the gradient of the function $\psi(s,t,u) - v$, is given by*

$$\begin{aligned}
\mathcal{L}(\psi) = &(\psi_{ss} + \psi_{tt})(1 + \psi_u^2) + \psi_{uu}(\psi_s^2 + \psi_t^2) \\
&+ 2\psi_{su}(\psi_t - \psi_u\psi_s) - 2\psi_{tu}(\psi_s + \psi_u\psi_t).
\end{aligned} \tag{1.8}$$

Now return to the case of general n. Denote coordinates on $\mathbb{C}^= = \mathbb{C}^{n-1} \oplus \mathbb{C}$ by (z, w) with $w = u + iv$. Let $\{v = \psi(z, u)\}$ be a hypersurface in $\mathbb{C}^n$ written as a graph. For a unit vector $a \in \mathbb{C}^{n-1}$ consider the function

$$\psi^a(\zeta, u) := \psi(\zeta a_1, \dots, \zeta a_{n-1}, u)$$

on $\mathbb{C} \times \mathbb{R}$. By Lemma 1.14 and Lemma 1.15, the hypersurface $\{v = \psi(z, u)\}$ is i-convex from above iff $\mathcal{L}(\psi^a) > 0$ for all unit vectors $a \in \mathbb{C}^{n-1}$. Note that the term $\psi^a_{ss} + \psi^a_{tt}$ is the Hermitian form H_{ψ^a} associated to the function $\zeta \mapsto \psi^a(\zeta, u)$ for constant u, which in turn equals the restriction of the Hermitian form H_ψ of ψ (for constant u) to the complex line $\mathbb{C}a$:

$$\psi^a_{ss} + \psi^a_{tt} = 4\sum_{i,j=1}^{n-1} \psi_{z_i\bar z_j} a_i \bar a_j = H_\psi(a, a).$$

On the other hand, from

$$\psi^a_s = \sum_j (\psi_{z_j} a_j + \psi_{\bar z_j}\bar a_j), \qquad \psi^a_t = i\sum_j (\psi_{z_j} a_j - \psi_{\bar z_j}\bar a_j)$$

we obtain the estimates

$$|\psi^a_s|, |\psi^a_t| \le |d_z\psi|, \quad |\psi^a_{su}|, |\psi^a_{tu}| \le |d_z\psi_u|.$$

Inserting these in the expression for the Levi form in Lemma 1.15 yields

Lemma 1.16. *A sufficient condition for the i-convexity of the hypersurface* $\{v = \psi(z, u)\} \subset \mathbb{C}^n$*, cooriented by the gradient of the function* $\psi(z, u) - v$*, is given by*

$$\mathcal{L}^{\min}(\psi) := H_\psi^{\min} - 2|\psi_{uu}|\,|d_z\psi|^2 - 4|d_z\psi_u|\,|d_z\psi|(1 + |\psi_u|) > 0,$$

where $H_\psi^{\min} := \min_{|a|=1} H_\psi(a, a)$.

Chapter 2

Smoothing

2.1 J-convexity and plurisubharmonicity

A C^2-function $\phi : U \to \mathbb{R}$ on an open domain $U \subset \mathbb{C}$ is i-convex if and only if it is *(strictly) subharmonic*, i.e.,

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 4\frac{\partial\phi}{\partial z\partial\bar{z}} > 0.$$

Note. By "subharmonic" we will always mean "strictly subharmonic". Non-strict subharmonicity will be referred to as "weak subharmonicity". The same applies to plurisubharmonicity discussed below.

A continuous function $\phi : U \to \mathbb{R}$ is called (strictly) subharmonic if it satisfies

$$\Delta\phi \geq h$$

for a positive continuous function $h : U \to \mathbb{R}$, where the Laplacian and the inequality are understood in the distributional sense, i.e.,

$$\int_U \phi\,\Delta\delta\;dx\,dy \geq \int_U h\,\delta\;dx\,dy \tag{2.1}$$

for any nonnegative smooth function $\delta : U \to \mathbb{R}$ with compact support. Note that if ϕ is a C^2-function satisfying (2.1), then integration by parts and choice of a sequence of functions δ_n converging to the Dirac measure of a point $p \in U$ shows $\Delta\phi(p) \geq h(p)$, so the two definitions agree for C^2-functions.

If $z = x + iy \to w = u + iv$ is a biholomorphic change of coordinates on U, then

$$\Delta_z\delta\;dx\,dy = 2i\frac{\partial^2\delta}{\partial z\partial\bar{z}}dz \wedge d\bar{z} = -dd^{\mathbb{C}}\delta = \Delta_w\delta\;du\,dv, \tag{2.2}$$

so inequality (2.1) transforms into

$$\int_U \phi(w)\Delta\delta(w)du\,dv \geq \int_U h(w)\delta(w)\Big|\frac{dz}{dw}\Big|^2du\,dv.$$

This shows that subharmonicity is invariant under biholomorphic coordinate changes and therefore can be defined for continuous functions on Riemann surfaces. The following lemma gives a useful criterion for subharmonicity of continuous functions.

Lemma 2.1. *A continuous function $\phi : U \to \mathbb{R}$ on a domain $U \subset \mathbb{C}$ is subharmonic if and only if there exists a positive continuous function $h : U \to \mathbb{R}$ such that*

$$\phi(z) + h(z)r^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(z + re^{i\theta})d\theta \tag{2.3}$$

for all $z \in U$ and $r > 0$ for which the disk of radius r around z in contained in U.

Proof. In a neighborhood of a point $z_0 \in U$ inequality (2.3) holds with h replaced by some constant $\lambda > 0$. Consider the function

$$\psi(z) := \phi(z) - \lambda|z - z_0|^2.$$

For $r > 0$ sufficiently small, (2.3) is equivalent to

$$\psi(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} \psi(z_0 + re^{i\theta})d\theta.$$

By a standard result (see e.g. [30]), this inequality is equivalent to $\Delta\psi(z_0) \geq 0$ in the distributional sense, and therefore to $\Delta\phi(z_0) \geq 2\lambda$.

□

Remark 2.2. The preceding proof shows: If ϕ in Lemma 2.1 is C^2, then inequality (2.3) holds with $h(z) := \frac{1}{4}\min_{D(z)}\Delta\phi$, where D is the maximal disk around z contained in U.

Now let (V, J) be an almost complex manifold. A *(nonsingular) J-holomorphic curve* is a 1-dimensional complex submanifold of (V, J). Note that the restriction of the almost complex structure J to a J-holomorphic curve is always integrable.

Lemma 2.3. *A C^2-function ϕ on an almost complex manifold (V, J) is J-convex if and only if its restriction to every J-holomorphic curve is subharmonic.*

Proof. By definition, ϕ is J-convex iff $-dd^{\mathbb{C}}\phi(X, JX) > 0$ for all $0 \neq X \in T_xV$, $x \in V$. Now for every such $X \neq 0$ there exists a J-holomorphic curve $C \subset V$ passing through x with $T_xC = span_{\mathbb{R}}\{X, JX\}$ ([39]). By formula (2.2) above, $-dd^{\mathbb{C}}\phi(X, JX) > 0$ precisely if $\phi|_C$ is subharmonic in x. □

Remark 2.4. In the proof we have used the fact that the differential operator $dd^{\mathbb{C}}$ commutes with restrictions to complex submanifolds. This is true because the exterior derivative and the composition with J both commute with restrictions to complex submanifolds.

Remark 2.5. Lemma 2.3 provides another proof of Corollary 5.9, i.e. that *non-degenerate critical points of a J-convex function have Morse indices $\leq n$.* Indeed, p be a critical point of a J-convex function ϕ. Suppose $\mathrm{ind}(p) > n$. Then there exists a subspace $W \subset T_pV$ of dimension $> n$ on which the Hessian of ϕ is negative definite. Since $W \cap JW \neq \{0\}$, W contains a complex line L. Let C be a J-holomorphic curve through p tangent to L. Then $\phi|_C$ attains a local maximum at p. But this contradicts the maximum principle because $\phi|_C$ is subharmonic by Lemma 2.3.

In view of Lemma 2.3 we can speak about *continuous J-convex functions* on almost complex manifolds as functions whose restrictions to all J-holomorphic curves are subharmonic. Such functions are also called *(strictly) plurisubharmonic*. For functions on $\mathbb{C}^n$, Lemma 2.1 and the proof of Lemma 2.3 show

Lemma 2.6. *A continuous function $\phi : \mathbb{C}^n \supset U \to \mathbb{R}$ is i-convex if and only if its restriction to every complex line is subharmonic. This means that there exists a positive continuous function $h : U \to \mathbb{R}$ such that*

$$\phi(z) + h(z)|w|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(z + we^{i\theta})d\theta \tag{2.4}$$

for all $z \in U$ and $w \in \mathbb{C}^n$ for which the disk of radius $|w|$ around z in contained in U.

The following lemma follows from equation (2.1) via integration by parts.

Lemma 2.7. *If ϕ is a J-convex function on an almost complex manifold (V, J), then $\phi + \psi$ is J-convex for every sufficiently C^2-small C^2-function $\psi : V \to \mathbb{R}$.*

Our interest in continuous J-convex functions is motivated by the following

Lemma 2.8. *If ϕ and ψ are continuous J-convex functions on an almost complex manifold (V, J), then $\max(\phi, \psi)$ is J-convex. More generally, let $(\phi_\lambda)_{\lambda\in\Lambda}$ be a continuous family of continuous functions, parameterized by a compact metric space Λ, that are uniformly J-convex in the sense that on every J-holomorphic disk $U \subset V$ condition* (2.3) *holds for all ϕ_λ with functions h_λ depending continuously on λ. Then $\max_{\lambda\in\Lambda}\phi_\lambda$ is a continuous J-convex function.*

Proof. Continuity of $\max_{\lambda\in\Lambda}\phi_\lambda$ is an easy exercise. For J-convexity we use the criterion from Lemma 2.1. Let $U \subset V$ be a J-holomorphic disk and choose a local coordinate z on U. By hypothesis, condition (2.3) holds for all ϕ_λ with functions h_λ depending continuously on λ. Note that $h(z) := \min_{\lambda\in\Lambda}h_\lambda$ defines a positive continuous function on U. Set $\phi := \max_{\lambda\in\Lambda}\phi_\lambda$. At any point $z \in U$ we have $\phi = \phi_\lambda$ for some $\lambda \in \Lambda$ (depending on z). Now the lemma follows from

$$\begin{aligned}\phi(z) + h(z)r^2 \leq \phi_\lambda(z) + h_\lambda(z)r^2 &\leq \frac{1}{2\pi}\int \phi_\lambda(z + re^{i\theta})d\theta \\ &\leq \frac{1}{2\pi}\int \phi(z + re^{i\theta})d\theta.\end{aligned}$$

□

Remark 2.9. For example, the hypotheses of Lemma 2.8 are satisfied if all the J-convex functions ϕ_λ are C^2 and their first two derivatives depend continuously on λ. This follows immediately from the remark after Lemma 2.1.

2.2 Smoothing of J-convex functions

For integrable J, continuous J-convex functions can be approximated by smooth ones. The following proposition was proved by Richberg [41]. We give below a proof following [15].

Proposition 2.10. *Let ϕ be a continuous J-convex function on a (integrable) complex manifold (V, J). Then for every positive function $h : V \to \mathbb{R}_+$ there exists a smooth J-convex function $\psi : V \to \mathbb{R}$ such that $|\phi(x) - \psi(x)| < h(x)$ for all $x \in V$. If ϕ is already smooth on a neighbourhood of a compact subset K, then we can achieve $\psi = \phi$ on K.*

Remark 2.11. A continuous weakly J-convex function cannot in general be approximated by smooth weakly J-convex functions, see [15] for a counterexample. We do not know whether the proposition remains true for almost complex manifolds.

The proof is based on an explicit smoothing procedure for functions on $\mathbb{R}^m$. Pick a smooth nonnegative function $\rho : \mathbb{C}^m \to \mathbb{R}$ with support in the unit ball and $\int_{\mathbb{R}^m} \rho = 1$. For $\delta > 0$ set $\rho_\delta(x) := \delta^{-m}\rho(x/\delta)$. Let $U \subset \mathbb{R}^m$ be an open subset and set

$$U_\delta := \{x \in U \mid \bar{B}_\delta(x) \subset U\}$$

For a continuous function $\phi : \mathbb{R}^m \supset U \to \mathbb{R}$ define the *mollified function* $\phi_\delta : U_\delta \to \mathbb{R}$,

$$\phi_\delta(x) := \int_{\mathbb{C}^n} \phi(x-y)\rho_\delta(y) d^{2n}y = \int_{\mathbb{C}^n} \phi(y)\rho_\delta(x-y) d^{2n}y. \tag{2.5}$$

The last expression shows that the functions ϕ_δ are smooth for every $\delta > 0$. The first expression shows that if ϕ is of class C^k for some $k \geq 0$, then $\phi_\delta \to \phi$ as $\delta \to 0$ uniformly on compact subsets of U.

Proposition 2.10 is an immediate consequence of the following lemma, via induction over a countable coordinate covering.

Lemma 2.12. *Let ϕ be a continuous J-convex function on a complex manifold (V, J). Let $A, B \subset V$ be compact subsets such that ϕ is smooth on a neighbourhood of A and B is contained in a holomorphic coordinate neighbourhood. Then for every $\varepsilon > 0$ and every neighbourhood W of $A \cup B$ there exists a continuous J-convex function $\psi : V \to \mathbb{R}$ with the following properties.*

- *ψ is smooth on a neighbourhood of $A \cup B$;*
- *$|\psi(x) - \phi(x)| < \varepsilon$ for all $x \in V$;*

- *$\psi = \phi$ on A and outside W.*

Proof. The proof follows [15]. First suppose that ϕ is i-convex on an open set $U \subset \mathbb{C}^n$. By Lemma 2.6, there exists a positive continuous function $h : U \to \mathbb{R}$ such that (2.4) holds for all $z \in U_{2\delta}$ and $w \in \mathbb{C}^n$ with $|w| \leq \delta$. Hence the mollified function ϕ_δ satisfies

$$\begin{aligned}\phi_\delta(x) + h_\delta(x)|w|^2 &= \int_{\mathbb{C}^n} \Big(\phi(x-y) + \delta(x-y)|w|^2\Big)\rho_\delta(y)d^{2n}y \\ &\leq \int_{\mathbb{C}^n}\int_0^{2\pi} \phi(x-y+we^{i\theta})d\theta\rho_\delta(y)d^{2n}y \\ &= \int_0^{2\pi} \phi_\delta(x+we^{i\theta})d\theta,\end{aligned}$$

so ϕ_δ is i-convex on $U_{2\delta}$.

Now let $\phi : V \to \mathbb{R}$ be as in the proposition. Pick a holomorphic coordinate neighbourhood U and compact neighbourhoods $A' \subset W$ of A and $B' \subset B'' \subset W \cap U$ of B with $A \subset \operatorname{int} A' \subset A' \subset W$, such that ϕ is smooth on A'. By the preceding discussion, there exists a smooth J-convex function $\phi_\delta : B'' \to \mathbb{R}$ with $|\phi_\delta(x) - \phi(x)| < \varepsilon/2$ for all $x \in B''$. Pick smooth cutoff functions $g, h : V \to [0,1]$ such that $g = 1$ on A, $g = 0$ outside A', $h = 1$ on B', and $h = 0$ outside B''. Define a continuous function $\tilde{\phi} : V \to \mathbb{R}$,

$$\tilde{\phi} := \phi + (1-g)h(\phi_\delta - \phi).$$

The function $\tilde{\phi}$ is smooth on $A' \cup B'$, $|\tilde{\phi}(x) - \phi(x)| < \varepsilon/2$ for all $x \in V$, $\tilde{\phi} = \phi_\delta$ on $B' \setminus A'$, and $\tilde{\phi} = \phi$ on A and outside B''. Since ϕ is C^2 on $A' \cap B''$, the function $(1-g)h(\phi_\delta - \phi)$ becomes arbitrarily C^2-small on this set for δ small. Hence by Lemma 2.7, $\tilde{\phi}$ is J-convex on $A' \cap B''$ for δ sufficiently small. So we can make $\tilde{\phi}$ J-convex on $A' \cup B'$. However, $\tilde{\phi}$ need not be J-convex on $B'' \setminus (A' \cup B')$.

Pick a compact neighbourhood $W' \subset W$ of $A' \cup B''$. Without loss of generality we may assume that ε was arbitrarily small. Then by Lemma 2.7 there exists a continuous J-convex function $\tilde{\psi} : V \to \mathbb{R}$ (which differs from ϕ by a C^2-small function) satisfying $\tilde{\psi} = \phi - \varepsilon$ on $A \cup B$, $\tilde{\psi} = \phi + \varepsilon$ on $W' \setminus (A' \cup B')$, and $\tilde{\psi} = \phi$ outside W. Now the function $\psi := \max(\tilde{\phi}, \tilde{\psi})$ has the desired properties. $\square$

Remark 2.13. The proof of Lemma 2.12 shows the following additional properties in Proposition 2.10:

(1) If ϕ_λ is a continuous family of J-convex functions depending on a parameter λ in a compact space Λ, then the ϕ_λ can be uniformly approximated by a continuous family of smooth J-convex functions ψ_λ.

(2) If $\phi_0 \leq \phi_1$ then the smoothed functions also satisfy $\psi_0 \leq \psi_1$. This holds because the proof only uses mollification $\phi \mapsto \phi_\delta$, interpolation and taking the maximum of two functions, all of which are monotone operations.

Lemma 2.8, the remark after it and Proposition 2.10 imply

Corollary 2.14. *The maximum of two smooth J-convex functions ϕ, ψ on a complex manifold (V, J) can be C^0-approximated by smooth J-convex functions. If* $\max(\phi, \psi)$ *is smooth on a neighbourhood of a compact subset K, then we can choose the approximating sequence to be equal to* $\max(\phi, \psi)$ *on K.*

More generally, let $(\phi_\lambda)_{\lambda\in\Lambda}$ be a continuous family of J-convex C^2-functions whose first two derivatives depend continuously on λ in a compact metric space Λ. Then $\max_{\lambda\in\Lambda}\phi_\lambda$ *can be C^0-approximated by smooth J-convex functions. If* $\max_{\lambda\in\Lambda}\phi_\lambda$ *is smooth on a neighbourhood of a compact subset K, then we can choose the approximating sequence to be equal to* $\max_{\lambda\in\Lambda}\phi_\lambda$ *on K.*

Finally, we show that we can arbitrarily prescribe a J-convex function near a totally real submanifold.

Proposition 2.15. *Let L be a totally real submanifold of a complex manifold (V, J) and $K \subset L$ a compact subset. Suppose that two smooth J-convex functions ϕ, ψ coincide along L together with their differentials, i.e. $\phi(x) = \psi(x)$ and $d\phi(x) = d\psi(x)$ for all $x \in L$. Then, given any neighbourhood U of K in V, there exists a J-convex function ϑ which coincides with ϕ outside U and with ψ in a smaller neighbourhood $U' \subset U$ of K. Moreover, ϑ can be chosen arbitrarily C^1-close to ϕ and such that ϑ agrees with ϕ together with its differential along L.*

The proof uses the following simple lemma. Consider an almost complex manifold (V, J) equipped with a Hermitian metric. To a smooth function $\phi : V \to \mathbb{R}$ we associate its *modulus of J-convexity* $m_\phi : V \to \mathbb{R}$,

$$m_\phi(x) := \min\{-dd^{\mathbb{C}}\phi(v, Jv) \mid v \in T_xV,\ |v| = 1\}.$$

Thus ϕ is J-convex iff $m_\phi > 0$.

Lemma 2.16. *Let $\phi, \psi, \beta : V \to \mathbb{R}$ be smooth functions on an almost complex manifold (V, J) such that*

$$|\phi(x) - \psi(x)|\ |dd^{\mathbb{C}}_x\beta| + 2|d_x\beta|\ |d_x(\phi - \psi)| < \min\big(m_\phi(x), m_\psi(x)\big)$$

for all $x \in V$ (with respect to some Hermitian metric). Then $(1 - \beta)\phi + \beta\psi$ is J-convex.

Proof. Adding up

$$dd^{\mathbb{C}}(\beta\psi) = \beta\, dd^{\mathbb{C}}\psi + d\beta \wedge d^{\mathbb{C}}\psi + d\psi \wedge d^{\mathbb{C}}\beta + \psi\, dd^{\mathbb{C}}\beta$$

and the corresponding equation for $(1 - \beta)\phi$ at any point $x \in V$, we find

$$\begin{aligned} -dd^{\mathbb{C}}\big((1-\beta)\phi + \beta\psi\big) &= -(1-\beta)\, dd^{\mathbb{C}}\phi - \beta dd^{\mathbb{C}}\psi + d\beta \wedge d^{\mathbb{C}}(\phi - \psi) \\ &\quad + d(\phi-\psi) \wedge d^{\mathbb{C}}\beta + (\phi - \psi)dd^{\mathbb{C}}\beta \\ &\geq \min(m_\phi, m_\psi) - 2|d\beta|\ |d(\phi-\psi)| - |\phi - \psi|\ |dd^{\mathbb{C}}\beta| \\ &> 0. \end{aligned}$$

□

Proof of Proposition 2.15. Fix a compact neighbourhood $\tilde{K} \subset L \cap U$ of K in L. Pick a Hermitian metric on (V, J) and consider the function dist_L^2, square of the distance to L, defined on a tubular neighborhood of L. According to Proposition 1.9, this function is J-convex. Hence

$$\phi_\lambda := \phi + \lambda \mathrm{dist}_L^2$$

is J-convex for any $\lambda \geq 0$. Since ϕ and ψ agree up to first order along L, there exists a $\lambda > 0$ and a compact neighbourhood $W \subset U$ of $\tilde{K}$ such that

$$\phi_\lambda > \psi \text{ on } W \setminus L.$$

For any $\varepsilon > 0$ we can find a $\delta < \varepsilon$ and a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\alpha(r) = r \text{ for } r \in [0, \delta], \quad \alpha(r) = 0 \text{ for } r \geq \varepsilon, \quad -\frac{3\delta}{\varepsilon} \leq \alpha' \leq 1, \quad |\alpha''| \leq \frac{3}{\varepsilon}.$$

Set

$$\tilde{\phi} := \phi + \lambda\alpha(\mathrm{dist}_L^2).$$

Then $\tilde{\phi}$ coincides with ϕ on $W \setminus U_\epsilon$ and with ϕ_λ on $W \cap U_\delta$, where $U_\varepsilon := \{\mathrm{dist}_L < \varepsilon\}$ denotes the ε-neighbourhood of L. Let us show that $\tilde{\phi}$ is J-convex. Indeed,

$$dd^{\mathbb{C}}\tilde{\phi} = dd^{\mathbb{C}}\phi + \lambda\alpha'' d(\mathrm{dist}_L^2) \wedge d^{\mathbb{C}}(\mathrm{dist}_L^2) + \lambda\alpha' dd^{\mathbb{C}}(\mathrm{dist}_L^2).$$

On $W \cap U_\varepsilon$ we have $|d(\mathrm{dist}_L^2)| \leq C\varepsilon$, where the constant C depends only on the geometry of $L \cap W$. Since $d(\mathrm{dist}_L^2) \wedge d^{\mathbb{C}}(\mathrm{dist}_L^2)$ is a quadratic function of $d(\mathrm{dist}_L^2)$, the second term on the right hand side can be estimated by

$$|\lambda\alpha'' d(\mathrm{dist}_L^2) \wedge d^{\mathbb{C}}(\mathrm{dist}_L^2)| \leq C_1\lambda \cdot \frac{1}{\varepsilon} \cdot \varepsilon^2$$

for some constant C_1. The third term on the right hand side is estimated by

$$\lambda\alpha' dd^{\mathbb{C}}(\mathrm{dist}_L^2) \geq -\lambda\frac{3\delta}{\varepsilon}|dd^{\mathbb{C}}(\mathrm{dist}_L^2)| \geq -\frac{C_2\lambda\delta}{\varepsilon}$$

for some constant C_2. Thus the modulus of J-convexity of $\tilde{\phi}$ satisfies

$$m_{\tilde{\phi}} \geq m_\phi - C_1\lambda\varepsilon - C_2\lambda\delta/\varepsilon.$$

So if $a := \min_W m_\phi > 0$, then $m_{\tilde{\phi}} \geq a/2 > 0$ on W whenever ε and δ/ε are sufficiently small.

Note that $\tilde{\phi}$ is arbitrarily C^1-close to ϕ for ε small. Fix a cutoff function β with support in W and equal to 1 on a neighbourhood $W' \subset W$ of $\tilde{K}$. The function

$$\bar{\phi} := (1 - \beta)\phi + \beta\tilde{\phi}.$$

satisfies $\bar{\phi} = \phi$ outside W and on L, and $\bar{\phi} > \psi$ on $W' \setminus L$. Moreover, since the estimates $m_\phi \geq a$ and $m_{\tilde{\phi}} \geq a/2$ are independent of ε and δ, Lemma 2.16 implies that $\bar{\phi}$ is J-convex if ε and δ/ε are sufficiently small.

Next pick a cutoff function γ with support in a smaller neighbourhood $W'' \subset W'$ of $\tilde{K}$ and equal to 1 near $\tilde{K}$. The function

$$\hat{\phi} := \bar{\phi} - \mu\gamma$$

is J-convex for $\mu > 0$ sufficiently small. Moreover, it satisfies

$$\hat{\phi} < \psi \text{ near } \tilde{K}, \qquad \hat{\phi} > \psi \text{ on } W' \backslash (W'' \cup L), \qquad \hat{\phi} = \phi \text{ outside } W, \qquad \hat{\phi} \le \psi \text{ on } L.$$

So the function

$$\hat{\vartheta} := \begin{cases} \max(\psi, \hat{\phi}) & \text{on } W', \\ \hat{\phi} & \text{outside } W' \end{cases}.$$

coincides with ψ near $\tilde{K}$ and on L and with ϕ outside W. Let $\tilde{\vartheta}$ be the J-convex function obtained by smoothing $\hat{\vartheta}$ as described in Corollary 2.14, leaving it unchanged near $\tilde{K}$ and outside W. Then $\tilde{\vartheta}$ coincides with ϕ outside W and with ψ near $\tilde{K}$. Moreover, since $\hat{\phi}$ is C^1-close to ϕ by construction, $\tilde{\vartheta}$ is C^1-close to ϕ by Corollary 2.23.

So $\tilde{\vartheta}$ has all the desired properties except that, due to the smoothing procedure, it may not agree with ϕ on $L \setminus \tilde{K}$. To remedy this, fix a cutoff function ρ with support in U which equals 1 near K and 0 on $L \setminus \tilde{K}$ and set

$$\vartheta := (1 - \rho)\phi + \rho\tilde{\vartheta}.$$

By Lemma 2.16, ϑ is J-convex if we choose $\tilde{\vartheta}$ sufficiently C^1-close to ϕ. Since ψ agrees with ϕ together with their differentials along L, the same holds for ϑ and ϕ. So ϑ is the desired function. □

The corresponding result for J-convex hypersurfaces is

Corollary 2.17. *Let Σ, Σ' be J-convex hypersurfaces in a complex manifold (V, J) that are tangent to each other along a totally real submanifold L. Then for any compact subset $K \subset L$ and neighbourhood U of K, there exists a J-convex hypersurface Σ'' that agrees with Σ outside U and with Σ' near K. Moreover, Σ'' can be chosen C^1-close to Σ and tangent to Σ along L.*

Proof. Pick smooth functions ϕ, ψ with regular level sets $\Sigma = \phi^{-1}(0)$ and $\Sigma' = \psi^{-1}(0)$ such that $d\phi = d\psi$ along L. By Lemma 1.3, after composing ϕ and ψ with the same convex function, we may assume that ϕ, ψ are J-convex on a neighbourhood $W \subset U$ of K. Let $\vartheta : W \to \mathbb{R}$ be the J-convex function from Proposition 1.9 which coincides with ψ near K and with ϕ outside a compact subset $W' \subset W$. Since ϑ is C^1-close to ϕ, it has 0 as a regular value and $\Sigma'' := \vartheta^{-1}(0)$ is the desired J-convex hypersurface. □

We will finish this section with the following

Lemma 2.18. *Let $\phi_0, \phi_1 : V \to \mathbb{R}_+$ be two exhausting J-convex functions. Then there exist smooth functions $h_0, h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ with $h_0, h_1' \to \infty$ and $h_0'', h_1'' > 0$, a completely exhausting function $\psi : V \to \mathbb{R}_+$, and a sequence of compact domains V_k, $k = 1, \dots,$ with smooth boundaries $\Sigma_k = \partial V_k$, such that*

- $V_k \subset \operatorname{Int} V_{k+1}$ *for all* $k \geq 1$;
- $\bigcup_k V_k = V$;
- Σ_{2j-1} *are level sets of the function* ϕ_1 *and* Σ_{2j} *are level sets of the function* $\phi_0, j = 1, \dots;$
- $\psi = h_0 \circ \phi_0$ *on* $\mathcal{O}p\left(\bigcup\limits_{j=1}^{\infty} \Sigma_{2j-1}\right)$ *and* $\psi = h_1 \circ \phi_1$ *on* $\mathcal{O}p\left(\bigcup\limits_{j=1}^{\infty} \Sigma_{2j}\right)$.

Proof of Lemma 2.18. We will call a diffeomorphism $h : \mathbb{R}^+ \to \mathbb{R}^+$ an *admissible function* if $h" \geq 0$ and $h' \to \infty$. Take any $c_1 > 0$, and denote $V_1 = \{\phi_1 \leq c_1\}, \Sigma_1 = \partial V_1$. There exists an admissible function g_1 such that $\phi_0|_{\Sigma_1} < d_1 = g_1(c_1)$.. Set $\psi_0 = \phi_0$, $\psi_1 = g_1 \circ \phi_1$. Take any $c_2 > d_1$ and denote $V_2 = \{\psi_0 \leq c_2\}, \Sigma_2 = \partial V_2$. Then $V_1 \subset \operatorname{Int} V_2$. There exists an admissible function g_2 such that $g_2(x) = x$ for $x \in [0, d_1]$ and $\psi_1|_{\Sigma_2} < d_2 = g_2(c_2)$. Set $\psi_2 = g_2 \circ \psi_0$. Continuing this process we will take $c_3 > d_2$ and denote $V_3 = \{\psi_1 \leq c_3\}, \Sigma_3 = \partial V_3$. There exists an admissible function g_3 such that $g_3(x) = x, x \in [0, d_2]$ and $\psi_2|_{\Sigma_3} < d_3 = g_3(c_3)$.. Set $\psi_3 = g_3 \circ \psi_1$, and so on. As a result of this process Continuing this process we construct two admissible functions h_0, h_1 and a sequence of compact domains V_k, $k = 1, \dots,$ such that

- $V_k \subset \operatorname{Int} V_{k+1}$ for all $k \geq 1$ and $\bigcup_k V_k = V$;
- ϕ_0 is constant on Σ_j for odd j, and ϕ_1 is constant on Σ_j if j is even;
- t $\psi_{\text{even}} = h_0 \circ \phi_0 = \lim\limits_{j \to \infty} \psi_{2j}$ and $\psi_{\text{odd}} = h_0 \circ \phi_1 = \lim\limits_{j \to \infty}$ psi_{2j-1};
- $\psi_1|_{\Sigma_{2j-1}} > \psi_0|_{\Sigma_{2j-1}}$, $\psi_0|_{\Sigma_{2j}} > \psi_1|_{\Sigma_{2j}}$ for all $j \geq 1$.

Then smoothing the continuous plurisubharmonic function $\max(\psi_0, \psi_1)$ we get the required smooth J-convex function ψ.

□

2.3 Critical points of J-convex functions

We wish to control the creation of new critical points under the construction of taking the maximum of two J-convex functions and then smoothing. This is based on the following trivial observation: A smooth function $\phi : M \to \mathbb{R}$ on a manifold has no critical points iff there exist a vector field X and a positive function h with $X \cdot \phi \geq h$. Multiplying by a nonnegative volume form Ω on M with compact support, we obtain

$$\int_M (X \cdot \phi)\Omega \geq \int_M h\Omega.$$

Using $(X \cdot \phi)\Omega + \phi L_X \Omega = L_X(\phi\Omega) = d(\phi i_X \Omega)$ and Stokes' theorem (assuming M is orientable over $\mathrm{supp}\Omega$), we can rewrite the left hand side as

$$\int_M (X \cdot \phi)\Omega = -\int_M \phi L_X \Omega.$$

So we have shown: A smooth function $\phi : M \to \mathbb{R}$ on a manifold has no critical points iff there exist a vector field X and a positive function h such that

$$-\int_M \phi L_X \Omega \geq \int_M h\Omega$$

for all nonnegative volume forms Ω on M with sufficiently small compact support. This criterion obviously still makes sense if ϕ is merely continuous. However, for technical reasons we will slightly modify it as follows.

We say that a continuous function $\phi : M \to \mathbb{R}$ satisfies $X \cdot \phi \geq h$ (in the distributional sense) if around each $p \in M$ there exists a coordinate chart $U \subset \mathbb{R}^m$ on which X corresponds to a constant vector field such that

$$-\int_U \phi L_X \Omega \geq h(p) \int_U \Omega$$

for all nonnegative volume forms Ω with support in U. Writing $\Omega = g(x)d^m x$ for a nonnegative function g, this is equivalent to

$$-\int_U \phi(x)(X \cdot g)(x)d^m x \geq h(p) \int_U g(x)d^m x. \tag{2.6}$$

This condition ensures that smoothing does not create new critical points:

Lemma 2.19. *If a continuous function $\phi : \mathbb{R}^m \supset U \to \mathbb{R}$ satisfies* (2.6) *for a constant vector field X and a constant $h = h(p) > 0$, then each mollified function ϕ_δ defined by equation* (2.5) *also satisfies* (2.6) *with the same X, h.*

Proof. Let g be a nonnegative test function with support in U and $0 < \delta < \mathrm{dist}(\mathrm{supp} g, \partial U)$. Let $y \in \mathbb{R}^m$ with $|y| < \delta$. Applying (2.6) to the function $x \mapsto g(x+y)$ and using translation invariance of X, h and the Lebesgue measure $dx := d^m x$, we find

$$\begin{aligned} -\int_U \phi(x-y)X \cdot g(x)dx &= -\int_U \phi(x)X \cdot g(x+y)dx \\ &\geq h \int_U g(x+y)dx = h \int_U g(x)dx. \end{aligned}$$

Multiplying by the nonnegative function ρ_δ and integrating yields

$$\begin{aligned} -\int_U \phi_\delta(x) X\cdot g(x)dx &= -\int_U\int_{B_\delta}\phi(x-y)\rho_\delta(y)X\cdot g(x)dy\,dx \\ &\geq h\int_U\int_{B_\delta} g(x)\rho_\delta(y)dy\,dx = h\int_U g(x)dx. \end{aligned}$$

□

The next proposition shows that the condition $X\cdot\phi\geq h$ is preserved under taking the maximum of functions.

Proposition 2.20. *Suppose the continuous functions $\phi,\psi : M\to\mathbb{R}$ satisfy $X\cdot\phi\geq h$, $X\cdot\psi\geq h$ with the same X,h. Then $X\cdot\max(\phi,\psi)\geq h$.*

More generally, suppose $(\phi_\lambda)_{\lambda\in\Lambda}$ is a continuous family of functions $\phi_\lambda : M\to\mathbb{R}$, parametrized by a compact separable metric space Λ, such that all ϕ_λ satisfy $X\cdot\phi_\lambda\geq h$ with the same X,h. Then $X\cdot\max_{\lambda\in\Lambda}\phi_\lambda\geq h$.

Proof. Let $U\subset\mathbb{R}^m$ be a coordinate chart and $X, h := h(p)$ be as in (2.6). After a rotation and rescaling, we may assume that $X=\frac{\partial}{\partial x_1}$. Suppose first that ϕ,ψ are smooth and 0 is a regular value of $\phi-\psi$. Then $\theta := \max(\phi,\psi)$ is a continuous function which is smooth outside the smooth hypersurface $\Sigma := \{x\in U \mid \phi(x)=\psi(x)\}$. Define the function $\frac{\partial\theta}{\partial x_1}$ as $\frac{\partial\phi(x)}{\partial x_1}$ if $\phi(x)\geq\psi(x)$ and $\frac{\partial\psi(x)}{\partial x_1}$ otherwise. We claim that $\frac{\partial\theta}{\partial x_1}$ is the weak x_1-derivative of θ. Indeed, for any test function g supported in U we have (orienting Σ as the boundary of $\{\phi\geq\psi\}$)

$$\begin{aligned} \int_U\frac{\partial\theta}{\partial x_1}g\,d^mx &= \int_{\{\phi\geq\psi\}}\frac{\partial\phi}{\partial x_1}g\,d^mx + \int_{\{\phi<\psi\}}\frac{\partial\psi}{\partial x_1}g\,d^mx \\ &= \int_\Sigma \phi g\,dx_2\dots dx_m - \int_{\{\phi\geq\psi\}}\phi\frac{\partial g}{\partial x_1}d^mx \\ &\quad - \int_\Sigma \psi g\,dx_2\dots dx_m - \int_{\{\phi<\psi\}}\frac{\partial\psi}{\partial x_1}g\,d^mx \\ &= -\int_U\theta\frac{\partial g}{\partial x_1}d^mx, \end{aligned}$$

since $\phi=\psi$ on Σ. This proves the claim. By hypothesis we have $\frac{\partial\theta}{\partial x}\geq h$, so the conclusion of the lemma follows via

$$-\int_U\theta\frac{\partial g}{\partial x_1}d^mx = \int_U\frac{\partial\theta}{\partial x_1}g\,d^mx \geq h\int_U g\,d^mx.$$

Next let $\phi,\psi : U\to\mathbb{R}$ be continuous functions satisfying (2.6). By Lemma 2.19, there exist sequences ϕ_k,ψ_k of smooth functions, converging locally uniformly to ϕ,ψ, such that $X\cdot\phi_k\geq h$ and $X\cdot\psi_k\geq h$ for all k. Perturb the ϕ_k to smooth functions $\tilde\phi_k$ such that 0 is a regular value of $\tilde\phi_k-\psi_k$, $\tilde\phi_k\to\phi$ locally uniformly, and $X \cdot \tilde\phi_k \geq h - 1/k$ for all k. By the smooth case above, the function $\max(\tilde\phi_k, \psi_k)$ satisfies

$$-\int_U \max(\tilde\phi_k, \psi_k) X \cdot g \, d^m x \geq (h - 1/k) \int_U g \, d^m x$$

for any nonnegative test function g supported in U. Since $\max(\tilde\phi_k, \psi_k) \to \max(\phi, \psi)$ locally uniformly, the limit $k \to \infty$ yields the conclusion of the lemma for two functions ϕ, ψ.

Finally, let $(\phi_\lambda)_{\lambda\in\Lambda}$ be a continuous family as in the lemma. Pick a dense sequence $\lambda_1, \lambda_2, \dots$ in Λ. Set $\psi_k := \max\{\phi_{\lambda_1}, \dots, \phi_{\lambda_k}\}$ and $\psi := \max_{\lambda\in\Lambda}\phi_\lambda$. By the lemma for two functions and induction, the functions ψ_k satisfy (2.6) with the same X, h for all k. Thus the lemma follows in the limit $k \to \infty$ if we can show locally uniform convergence $\psi_k \to \psi$.

We first prove pointwise convergence $\psi_k \to \psi$. So let $x \in U$. Then $\psi(x) = \phi_\lambda(x)$ for some $\lambda \in \Lambda$. Pick a sequence k_ℓ such that $\lambda_{k_\ell} \to \lambda$ as $\ell \to \infty$. Then $\phi_{\lambda_{k_\ell}}(x) \to \phi_\lambda(x) = \psi(x)$ as $\ell \to \infty$. Since $\phi_{\lambda_{k_\ell}}(x) \leq \psi_{k_\ell}(x) \leq \psi(x)$, this implies $\psi_{k_\ell}(x) \to \psi(x)$ as $\ell \to \infty$. Now the convergence $\psi_k(x) \to \psi(x)$ follows from monotonicity of the sequence $\psi_k(x)$.

So we have an increasing sequence of continuous functions ψ_k that converges pointwise to a continuous limit function ψ. By a simple argument this implies locally uniform convergence $\psi_k \to \psi$: Let $\varepsilon > 0$ and $x \in U$ be given. By pointwise convergence there exists a k such that $\psi_k(x) \geq \psi(x) - \varepsilon$. By continuity of ϕ_k and ψ, there exists a $\delta > 0$ such that $|\psi_k(y) - \psi_k(x)| < \varepsilon$ and $|\psi(y) - \psi(x)| < \varepsilon$ for all y with $|y - x| < \delta$. This implies $\psi_k(y) \geq \psi(y) - 3\varepsilon$ for all y with $|y - x| < \delta$. In view of monotonicity, this establishes locally uniform convergence $\psi_k \to \psi$ and hence concludes the proof of the proposition. □

Finally, we show that J-convex functions can be smoothed without creating critical points.

Proposition 2.21. *Let $\phi : V \to \mathbb{R}$ be a continuous J-convex function on a complex manifold satisfying $X \cdot \phi \geq h$ for a vector field X and a positive function $h : V \to \mathbb{R}$. Then the J-convex smoothing $\psi : V \to \mathbb{R}$ in Proposition 2.10 can be constructed so that it satisfies $X \cdot \psi \geq \tilde h$ for any given function $\tilde h < h$.*

Proof. The function ψ is constructed from ϕ in Lemma 2.12 by repeated application of the following 3 constructions:

(1) Mollification $\phi \mapsto \phi_\delta$. This operation preserves the condition $X \cdot \phi \geq h$ by Lemma 2.19.

(2) Taking the maximum of two functions. This operation preserves the condition $X \cdot \phi \geq h$ by Proposition 2.20.

(3) Adding a C^2-small function f to ϕ. Let $k : V \to \mathbb{R}$ be a small positive function such that $\sup_U (X \cdot f)(x) \geq -k(p)$ for each coordinate chart U around

p as in condition (2.6) (for this it suffices that f is sufficiently C^1-small). Then we find

$$-\int_U f(x)(X\cdot g)(x)dx = \int_U (X\cdot f)(x)g(x)dx \geq -k(p)\int_U g(x)dx,$$

so the function $\phi + f$ satisfies $X\cdot(\phi+f)\geq h-k$. In the proof of Lemma 2.12, this operation is applied finitely many times on each compact subset of V, so by choosing the function k sufficiently small we can achieve that $X\cdot\psi\geq\tilde{h}$. □

Remark 2.22. Inspection of the proofs shows that Propositions 2.20 and 2.21 remain valid if all inequalities are replaced by the reverse inequalities.

Corollary 2.23. *If two smooth J-convex functions ϕ,ψ on a complex manifold V are C^1-close, then the smoothing of* $\max(\phi,\psi)$ *is C^1-close to ϕ.*

Proof. Let X be a vector field and $h_\pm : V\to\mathbb{R}$ functions such that $h_-\leq X\cdot\phi, X\cdot\psi\leq h_+$. By the preceding remark, the smoothing ϑ of $\max(\phi,\psi)$ can be constructed such that $\tilde{h}_-\leq X\cdot\vartheta\leq\tilde{h}_+$ for any given functions $\tilde{h}_- < h_-$ and $\tilde{h}_+ > h_+$. Since X, h_-, h_+ were arbitrary, this proves C^1-closeness of ϑ to ϕ. □

Finally, we apply the preceding result to smoothing of J-convex hypersurfaces.

Corollary 2.24. *Let $(M\times\mathbb{R}, J)$ be a compact complex manifold and $\phi,\psi : M\to\mathbb{R}$ two functions whose graphs are J-convex cooriented by ∂_r, where r is the coordinate on $\mathbb{R}$. Then there exists a smooth function $\theta : M\to\mathbb{R}$ with J-convex graph which is C^0-close to* $\min(\phi,\psi)$ *and coincides with* $\min(\phi,\psi)$ *outside a neighbourhood of the set $\{\phi=\psi\}$.*

Proof. For a convex increasing function $f:\mathbb{R}\to\mathbb{R}$ with $f(0)=0$ consider the functions

$$\Phi(x,r) := f\big(r-\phi(x)\big), \qquad \Psi(x,r) := f\big(r-\psi(x)\big).$$

For f sufficiently convex, Φ and Ψ are J-convex and satisfy $\partial_r\Phi > 0, \partial_r\Psi > 0$ near their zero level sets. Thus by Propositions 2.20 and 2.21 the function $\max(\Phi,\Psi)$ can be smoothed, keeping it fixed outside a neighbourhood U of the set $\{\max(\Phi,\Psi)=0\}$, to a function Θ which is J-convex and satisfies $\partial_r\Theta > 0$ near its zero level set. The last condition implies that the smooth J-convex hypersurface $\Theta^{-1}(0)$ is the graph of a smooth function $\theta : M\to\mathbb{R}$. Now note that the zero level set $\{\max(\Phi,\Psi)=0\}$ is the graph of the function $\min(\phi,\psi)$. This implies that θ is C^0-close to $\min(\phi,\psi)$ and coincides with $\min(\phi,\psi)$ outside U. □

2.4 From families of hypersurfaces to J-convex functions

The following result shows that a continuous family of J-convex hypersurfaces transverse to the same vector field gives rise to a smooth function with regular J-convex level sets. This will be extremely useful for the construction of J-convex functions with prescribed critical points.

Proposition 2.25. *Let $(M\times[0,1],J)$ be a compact complex manifold such that $M\times\{0\}$ and $M\times\{1\}$ are J-convex cooriented by ∂_r, where r is the coordinate on $[0,1]$. Suppose there exists a smooth family $(\Sigma_\lambda)_{\lambda\in[0,1]}$ of J-convex hypersurfaces transverse to ∂_r with $\Sigma_0 = M\times\{0\}$ and $\Sigma_1 = M\times\{1\}$. Then there exists a smooth foliation $(\tilde\Sigma_\lambda)_{\lambda\in[0,1]}$ of $M\times[0,1]$ by J-convex hypersurfaces transverse to ∂_r with $\tilde\Sigma_\lambda = M\times\{\lambda\}$ for λ near 0 or 1.*

Proof. Let $\varepsilon>0$ be so small that the hypersurfaces $M\times\{\lambda\}$ are J-convex for $\lambda\le\varepsilon$ and $\lambda\ge 1-\varepsilon$. Set $V := M\times[0,1]$ and $U := M\times(\varepsilon,1-\varepsilon)$. Reparametrize in λ such that $\Sigma_\lambda = M\times\{\lambda\}$ for $\lambda\le\varepsilon$ and $\lambda\ge 1-\varepsilon$. After a C^2-small perturbation and decreasing ε, we may further assume that $\Sigma_\lambda\subset U$ for $\lambda\in(\varepsilon,1-\varepsilon)$. Pick a smooth family of J-convex functions ϕ_λ with regular level sets $\phi_\lambda^{-1}(0)=\Sigma_\lambda$. After composing each ϕ_λ with a suitable function $\mathbb{R}\to\mathbb{R}$, we may assume that $\phi_\lambda>\phi_\mu$ for all $\lambda<\mu$ with either $\lambda\le\varepsilon$ or $\mu\ge 1-\varepsilon$.

The continuous functions

$$\psi_\lambda := \max\nolimits_{\nu\ge\lambda}\phi_\nu$$

are J-convex by Lemma 2.8 and, by construction, satisfy

$$\psi_\lambda \ge \psi_\mu \text{ for } \lambda\le\mu. \tag{2.7}$$

Moreover, we have $\psi_\lambda=\phi_\lambda$ for $\lambda\le\varepsilon$ and $\lambda\ge 1-\varepsilon$. By Proposition 2.20, the ψ_λ satisfy $\partial_r\cdot\psi_\lambda\ge h$ (in the distributional sense) for a positive function $h: M\times[0,1]\to\mathbb{R}$.

Next use Proposition 2.10 to approximate the ψ_λ by smooth J-convex functions $\hat\psi_\lambda$. By Remark 2.13, the resulting family $\hat\psi_\lambda$ is continuous in λ and still satisfies (2.7). By Proposition 2.21, the smoothed functions satisfy $\partial_r\cdot\hat\psi_\lambda\ge h/2>0$, hence the level sets $\hat\Sigma_\lambda := \hat\psi_\lambda^{-1}(0)$ are regular and transverse to ∂_r. We can modify the smoothing construction to achieve $\hat\psi_\lambda=\phi_\lambda$ near $\lambda=0$ and 1, still satisfying J-convexity,transversality of the zero level to ∂_r, and (2.7). Note that as a result of the smoothing construction the functions $\hat\psi_\lambda$, and hence their level sets $\hat\Sigma_\lambda$, depend continuously on the parameter λ with respect to the C^2-topology.

Since $\hat\Sigma_\lambda$ is transverse to ∂_r, we can write it as the graph $\{r=f_\lambda(x)\}$ of a smooth function $f_\lambda: M\to[0,1]$. By construction, the functions f_λ depend continuously on λ with respect to the C^2-topology, $f_\lambda\le f_\mu$ for $\lambda\le\mu$, and $f_\lambda(x)=\lambda$ for $\lambda\le\varepsilon$ and $\lambda\ge 1-\varepsilon$, with some $\varepsilon>0$ (possibly smaller than the one above). Note that $f_\mu(x) - f_\lambda(x) \geq \mu - \lambda$ for $\lambda \leq \mu \leq \varepsilon$ and $1-\varepsilon \leq \lambda \leq \mu$. Pick a function $g : [0,1] \to [0,1]$ satisfying $g(\lambda) = 0$ for $\lambda \leq \varepsilon/2$ and $\lambda \geq 1-\varepsilon/2$, $g'(\lambda) \geq -1+\gamma$ for $\varepsilon/2 \leq \lambda \leq \varepsilon$ and $1-\varepsilon \leq \lambda \leq 1-\varepsilon/2$, and $g'(\lambda) \geq \gamma$ for $\varepsilon \leq \lambda \leq 1-\varepsilon$, with some $\gamma > 0$. For g sufficiently small, the graphs of the functions $\hat{f}_\lambda(x) := f_\lambda(x) + g(\lambda)$ are still J-convex, $\hat{f}_\lambda(x) = \lambda$ for $\lambda \leq \varepsilon/2$ and $\lambda \geq 1-\varepsilon/2$, and

$$\hat{f}_\mu(x) - \hat{f}_\lambda(x) \geq \gamma(\mu - \lambda)$$

for all $\lambda \leq \mu$. Now mollify the functions $\hat{f}_\lambda(x)$ in the parameter λ to

$$\tilde{f}_\lambda(x) := \int_{\mathbb{R}} \hat{f}_{\lambda-\mu}(x)\rho_\delta(\mu)d\mu,$$

with a cutoff function $\rho : \mathbb{R} \to \mathbb{R}$ as in equation (2.5). Since the functions $f_{\lambda-\mu}$ are C^2-close to f_λ for $\mu \in \operatorname{supp}(\rho_\delta)$ and δ small, the graph of $\tilde{f}_\lambda$ is C^2-close to the graph of f_λ and hence J-convex. Moreover, for $\lambda' \geq \lambda$ the $\tilde{f}_\lambda$ still satisfy

$$\tilde{f}_{\lambda'}(x) = \int_{\mathbb{R}} \hat{f}_{\lambda'-\mu}(x)\rho_\delta(\mu)d\mu \geq \int_{\mathbb{R}} \hat{f}_{\lambda-\mu}(x)\rho_\delta(\mu)d\mu + \gamma(\lambda'-\lambda) = \tilde{f}_\lambda(x) + \gamma(\lambda'-\lambda).$$

Modify the $\tilde{f}_\lambda$ such that $\tilde{f}_\lambda(x) = \lambda$ for $\lambda \leq \varepsilon/2$ and $\lambda \geq 1-\varepsilon/2$, and so that their graphs are still J-convex and $\tilde{f}_\mu(x) - \tilde{f}_\lambda(x) \geq \gamma(\mu-\lambda)$ for all $\lambda \leq \mu$. The last inequality implies that the map $(x,\lambda) \mapsto \big(x, f_\lambda(x)\big)$ is an embedding, thus the graphs of $\tilde{f}_\lambda$ form the desired foliation $\tilde{\Sigma}_\lambda$. □

Chapter 3

Symplectic and Contact Preliminaries

In this chapter we collect some relevant facts from symplectic and contact geometry. For more details see [**?**].

3.1 Symplectic vector spaces

A *symplectic vector space* (V, ω) is a (finite dimensional) vector space V with a nondegenerate skew-symmetric bilinear form ω. Here nondegenerate means that $v \mapsto \omega(v, \cdot)$ defines an isomorphism $V \mapsto V^*$. A linear map $\Psi : (V_1, \omega_1) \to (V_2, \omega_2)$ between symplectic vector spaces is called *symplectic* if $\Psi^*\omega_2 \equiv \omega_2(\Psi\cdot, \Psi\cdot) = \omega_1$.

For any vector space U the space $U \oplus U^*$ carries the *standard symplectic structure*

$$\omega_{\text{st}}\Big((u, u^*), (v, v^*)\Big) := v^*(u) - u^*(v).$$

In coordinates q_i on U and dual coordinates p_i on U^*, the standard symplectic form is given by

$$\omega_{\text{st}} = \sum dq_i \wedge dp_i.$$

Define the ω*-orthogonal complement* of a linear subspace $W \subset V$ by

$$W^\omega := \{v \in V \bigm| \omega(v, w) = 0 \text{ for all } w \in W\}.$$

Note that $\dim W + \dim W^\omega = 2n$, but $W \cap W^\omega$ need not be $\{0\}$. W is called

- *symplectic* if $W \cap W^\omega = \{0\}$;
- *isotropic* if $W \subset W^\omega$;

- *coisotropic* if $W^\omega \subset W$;
- *Lagrangian* if $W^\omega = W$.

Note that $\dim W$ is even for W symplectic, $\dim W \leq n$ for W isotropic, $\dim W \geq n$ for W coisotropic, and $\dim W = n$ for W Lagrangian. Note also that $(W^\omega)^\omega = W$, and $\big(W/(W \cap W^\omega), \omega\big)$ is a symplectic vector space.

Consider a subspace W of a symplectic vector space (V, ω) and set $N := W \cap W^\omega$. Choose subspaces $V_1 \subset W$, $V_2 \subset W^\omega$ and $V_3 \subset (V_1 \oplus V_2)^\omega$ such that

$$W = V_1 \oplus N, \quad W^\omega = N \oplus V_2, \quad (V_1 \oplus V_2)^\omega = N \oplus V_3.$$

Then the decomposition

$$V = V_1 \oplus N \oplus V_2 \oplus V_3$$

induces a symplectic isomorphism

$$\begin{aligned} (V, \omega) &\to (W/N, \omega) \oplus (W^\omega/N, \omega) \oplus (N \oplus N^*, \omega_{\rm st}), \\ v_1 + n + v_2 + v_3 &\mapsto \Big(v_1, v_2, (n, -i_{v_3}\omega)\Big). \end{aligned} \tag{3.1}$$

Every symplectic vector space (V, ω) of dimension $2n$ possesses a *symplectic basis* $e_1, f_1, \ldots, e_n, f_n$, i.e. a basis satisfying

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \qquad \omega(e_i, e_j) = \delta_{ij}.$$

Moreover, given a subspace $W \subset V$, the basis can be chosen such that

- $W = \mathrm{span}\{e_1, \ldots, e_{k+l}, f_1, \ldots, f_k\}$;
- $W^\omega = \mathrm{span}\{e_{k+1}, \ldots, e_n, f_{k+l+1}, \ldots, f_n\}$;
- $W \cap W^\omega = \mathrm{span}\{e_{k+1}, \ldots, e_{k+l}\}$.

In particular, we get the following normal forms:

- $W = \mathrm{span}\{e_1, f_1, \ldots, e_k, f_k\}$ if W is symplectic;
- $W = \mathrm{span}\{e_1, \ldots, e_k\}$ if W is isotropic;
- $W = \mathrm{span}\{e_1, \ldots, e_n, f_1, \ldots, f_k\}$ if W is coisotropic;
- $W = \mathrm{span}\{e_1, \ldots, e_n\}$ if W is Lagrangian.

This reduces the study of symplectic vector spaces to the *standard symplectic space* $(\mathbb{R}^{2n}, \omega_{\rm st} = \sum dq_i \wedge dp_i)$.

A pair (ω, J) of a symplectic form ω and a complex structure J on a vector space V is called *compatible* if

$$g_J := \omega(\cdot, J\cdot)$$

is an inner product (i.e. symmetric and positive definite). This is equivalent to saying that

$$H(v, w) := \omega(Jv, w) - i\omega(v, w)$$

defines a Hermitian metric. Therefore, we will also call a compatible pair (ω, J) a *Hermitian structure*.

Lemma 3.1. *(a) The space of symplectic forms compatible with a given complex structure is nonempty and contractible.*

(b) The space of complex structures compatible with a given symplectic form is nonempty and contractible.

Proof. (a) immediately follows from the fact that the Hermitian metrics for a given complex structure form a convex space.

(b) is a direct consequence of the following fact (see [**?**]): For a symplectic vector space (V, ω) there exists a continuous map from the space of inner products to the space of compatible complex structures which maps each induced inner product g_J to J.

To see this fact, note that an inner product g defines an isomorphism $A : V \to V$ via $\omega(\cdot, \cdot) = g(A\cdot, \cdot)$. Skew-symmetry of ω implies $A^T = -A$. Recall that each positive definite operator P possesses a unique positive definite square root $\sqrt{P}$, and $\sqrt{P}$ commutes with every operator with which P commutes. So we can define

$$J_g := (AA^T)^{-\frac{1}{2}} A.$$

It follows that $J_g^2 = -\mathbb{1}$ and $\omega(\cdot, J\cdot) = g(\sqrt{AA^T}\cdot, \cdot)$ is an inner product. Continuity of the mapping $g \mapsto J_g$ follows from continuity of the square root. Finally, if $g = g_J$ for some J then $A = J = J_g$. □

3.2 Symplectic vector bundles

The discussion of the previous section immediately carries over to vector bundles. For this, let $E \to M$ be a real vector bundle of rank $2n$ over a manifold. A *symplectic structure* on E is a smooth section ω in the bundle $\Lambda^2 E^* \to M$ such that each $\omega_x \in \Lambda^2 E_x^*$ is a linear symplectic form. A pair (ω, J) of a symplectic and a complex structure on E is called *compatible*, or a *Hermitian structure*, if $\omega(\cdot, J\cdot)$ defines an inner product on E. Lemma **??** immediately yields the following facts, where the spaces of sections are equipped with any reasonable topology, e.g. the C^∞_{loc} topology:

(a) The space of compatible complex structures on a symplectic vector bundle (E, ω) is nonempty and contractible.

(b) The space of compatible symplectic structures on a complex vector bundle (E, J) is nonempty and contractible.

This shows that the homotopy theories of symplectic, complex and Hermitian vector bundles are the same. In particular, obstructions to trivialization of a symplectic vector bundle (E, ω) are measured by the *Chern classes* $c_k(E, \omega) = c_k(E, J)$ for any compatible complex structure J.

Remark 3.2. The homotopy equivalence between symplectic, complex and Hermitian vector bundles can also be seen in terms of their structure groups: The symplectic group [1]

$$Sp(2n) := \{\Psi \in GL(2n, \mathbb{R}) \mid \Psi^* \omega = \omega\} = \{\Psi \in GL(2n, \mathbb{R}) \mid \Psi^T J \Psi = J\}$$

and the general complex linear group $GL(n, \mathbb{C})$ both deformation retract onto the unitary group

$$U(n) = Sp(2n) \cap O(2n) = O(2n) \cap GL(n, \mathbb{C}) = GL(n, \mathbb{C}) \cap Sp(2n).$$

We end this section with a normal form for subbundles of symplectic vector bundles.

Proposition 3.3. *Let (E, ω) be a rank $2n$ symplectic vector bundle and $W \subset E$ a rank $2k + l$ subbundle such that $N := W \cap W^\omega$ has constant rank l. Then*

$$(E, \omega) \cong (W/N, \omega) \oplus (W^\omega/N, \omega) \oplus (N \oplus N^*, \omega_{\mathrm{st}}).$$

Proof. Pick a compatible almost complex structure J on (E, ω). Then

$$V_1 := W \cap JW, \quad V_2 := W^\omega \cap JW^\omega, \quad V_3 := JN$$

are smooth subbundles of E. Now the isomorphism 3.1 of the previous section yields the desired decomposition. □

3.3 Symplectic manifolds

A *symplectic manifold* (V, ω) is a manifold V with a closed nondegenerate 2-form ω. A map $f : (V_1, \omega_1) \to (V_2, \omega_2)$ between symplectic manifolds is called *symplectic* if $f^* \omega_2 = \omega_1$, and a symplectic diffeomorphism is called *symplectomorphism*. The following basic results states that every symplectic manifold of dimension $2n$ is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$. In other words, every symplectic manifold possesses a *symplectic atlas*, i.e. an atlas all of whose transition maps are symplectic.

Proposition 3.4 (symplectic Darboux Theorem)**.** *Let (V, ω) be a symplectic manifold of dimension $2n$. Then every $x \in V$ possesses a coordinate neighbourhood U and a coordinate map $\phi : U \to U' \subset \mathbb{R}^{2n}$ such that $\phi^* \omega_{st} = \omega$.*

[1] $Sp(2n)$ is *not* the "symplectic group" $Sp(n)$ considered in Lie group theory. E.g., the latter is compact, while our symplectic group is not.

The symplectic Darboux Theorem is a special case of the Symplectic Neighbourhood Theorem which will be proved in the next section. Now let us discuss some examples of symplectic manifolds.

Cotangent bundles. Let $T^*Q \xrightarrow{\pi} Q$ be the cotangent bundle of a manifold Q. The 1-form $\sum p_i dq_i$ is independent of coordinates q_i on Q and dual coordinates p_i on T_q^*Q and thus defines the *Liouville 1-form* $\lambda_{\rm st}$ on T^*Q. Intrinsically,

$$(\lambda_{\rm st})_{(q,p)} \cdot v = \langle p, T_{(q,p)}\pi \cdot v \rangle \quad \text{for } v \in T_{(q,p)}T^*Q,$$

where $\langle \ , \ \rangle$ is the pairing between T_q^*Q and T_qQ. The 2-form $\omega_{\rm st} := -d\lambda_{\rm st}$ is clearly closed, and the coordinate expression $\omega_{\rm st} = \sum dq_i \wedge dp_i$ shows that it is also nondegenerate. So $\omega_{\rm st}$ defines the *standard symplectic form* on T^*Q. The standard form on $\mathbb{R}^{2n}$ is a particular case of this construction.

Sign?

Almost complex submanifolds. A pair (ω, J) of a symplectic form and an almost complex structure on V is called *compatible* if $\omega(\cdot, J\cdot)$ defines a Riemannian metric. It follows that ω induces a symplectic form on every almost complex submanifold $W \subset V$ (which is compatible with $J|_W$).

J-convex functions. If (V, J) is an almost complex structure and $\phi : V \to \mathbb{R}$ a J-convex function, then the 2-form $\omega_\phi = -dd^{\mathbb{C}}\phi$ is symplectic. Moreover, ω_ϕ is compatible with J if J is integrable (see Section **??**). In particular, every J-convex function on a Stein manifold induces a symplectic form compatible with J.

Kähler manifolds. A Kähler manifold is a complex manifold (V, J) with a *Kähler metric*, i.e. a Hermitian metric $H = g - i\omega$ on TV such that the 2-form ω is closed. Thus the *Kähler form* ω is a symplectic form compatible with J. Note that every complex submanifold of a Kähler manifold is again Kähler.

The two basic examples of Kähler manifolds are $\mathbb{C}^n$ with the standard complex structure and Hermitian metric, and the complex projective space $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus 0)/(\mathbb{C} \setminus 0)$ with the induced complex structure and Hermitian metric (the latter is defined by restricting the Hermitian metric of $\mathbb{C}^{n+1}$ to the unit sphere and dividing out the standard circle action). Passing to complex submanifolds of $\mathbb{C}^n$, we see again that Stein manifolds are Kähler. Passing to complex submanifolds of $\mathbb{C}P^n$, we see that smooth projective varieties are Kähler. This gives us a rich source of examples of closed symplectic manifolds.

Remark 3.5. While cotangent bundles and Kähler manifolds provide obvious examples of symplectic manifolds, it is not obvious how to go beyond them. The first example of a closed symplectic manifold that is not Kähler was presented by Thurston in 1976 ([**?**]). In 1995 Gompf [**?**] proved that every finitely presented group is the fundamental group of a closed symplectic 4-manifold, in start contrast to the many restrictions on the fundamental groups of closed Kähler surfaces.

Problem 3.1. Show that a Riemannian metric g on a manifold Q induces a natural almost complex structure J_g on T^*Q, compatible with $\omega_{\rm st}$, which interchanges the horizontal and vertical subspaces defined by the Levi-Civita connection. Prove that J_g is integrable if and only if the metric g is flat.

3.4 Moser's trick and symplectic normal forms

An (embedded or immersed) submanifold W of a symplectic manifold (V, ω) is called *symplectic (isotropic, coisotropic, Lagrangian)* if $T_xW \subset T_xV$ is symplectic (isotropic, coisotropic, Lagrangian) for every $x \in W$ in the sense of Section 3.1. In this section we derive normal forms for neighbourhoods of such submanifolds.

All the normal forms can be proved by the same technique which we will refer to as *Moser's trick*. It is based on Cartan's formula $L_X\alpha = i_X d\alpha + d\, i_X\alpha$ for a vector field X and a k-form α. Suppose we are given k-forms α_0, α_1 on a manifold M, and we are looking for a diffeomorphism $\phi : M \to M$ such that $\phi^*\alpha_1 = \alpha_0$. Moser's trick is to construct ϕ as the time-1 map of a time-dependent vector field X_t. For this, let α_t be a smooth family of k-forms connecting α_0 and α_1, and look for a vector field X_t whose flow ϕ_t satisfies

$$\phi_t^*\alpha_t \equiv \alpha_0. \tag{3.2}$$

Then the time-1 map $\phi = \phi_1$ the solves our problem. Now equation (3.2) follows by integration (provided the flow of X_t exists, e.g. if X_t has compact support) once its linearized version

$$0 = \frac{d}{dt}\phi_t^*\alpha_t = \phi_t^*(\dot\alpha_t + L_{X_t}\alpha_t)$$

holds for every t. Inserting Cartan's formula, this reduces the problem to the algebraic problem of finding a vector field X_t that satisfies

$$\dot\alpha_t + d\, i_{X_t}\alpha_t + i_{X_t}d\alpha_t = 0. \tag{3.3}$$

Need version for non-closed V.

Here is a first application of this method.

Theorem 3.6 (Moser's Stability Theorem). *Let ω_t be a smooth family of cohomologous symplectic forms on a closed manifold. Then there exists a smooth family of diffeomorphisms ϕ_t such that $\phi_t^*\omega_t = \omega_0$.*

Proof. Since the ω_t are cohomologous, $\dot\omega_t$ is trivial in cohomology, so for every t there exists a 1-form β_t such that $d\beta_t = \dot\omega_t$. The formst β_t are not unique, but they can be chosen to depend smoothly on t. This can be achieved either by local arguments in coordinate charts (cf. [**?**], Theorem 3.17), or by Hodge theory as follows: Pick a Riemannian metric on the manifold V and let $d^* : \Omega^2(V) \to \Omega^1(V)$ be the L^2-adjoint of d. By Hodge theory, $\mathrm{im}(d^*) = \ker(d)^\perp$, so d is an isomorphism from $\mathrm{im}(d^*)$ to the exact 2-forms. The inverse of this isomorphism provides the particular choice for β_t.

Now we can solve equation (3.3),

$$0 = \dot\omega_t + d\, i_{X_t}\omega_t + i_{X_t}d\omega_t = d(\beta_t + i_{X_t}\omega_t)$$

by solving $\beta_t + i_{X_t}\omega_t = 0$, which has a unique solution X_t due to the nondegeneracy of ω_t. □

Our second application of Moser's trick is the following lemma, which is the basis of all the normal form theorems below.

Lemma 3.7. *Let W be a compact submanifold of a manifold V, and let ω_0, ω_1 be symplectic forms on V which agree at all points of W. Then there exist tubular neighbourhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = \mathbb{1}$ and $\phi^*\omega_1 = \omega_0$.*

Proof. Set $\omega_t := (1-t)\omega_0 + \omega_1$. Since $\omega_t \equiv \omega_0$ along W, ω_t are symplectic forms on some tubular neighbourhood U of W. By the relative de Rham Theorem, since $\dot\omega_t = \omega_1 - \omega_0$ is closed and vanishes along W, there exists a of 1-form β on U such that $\beta = 0$ along W and $d\beta = \dot\omega_t$ on U. As in the proof of Theorem 3.6, we solve equation (3.3) by setting $\beta + i_{X_t}\omega_t = 0$.

To apply Moser's trick, a little care is needed because U is noncompact, so the flow of X_t may not exist until time 1. However, since $\beta = 0$ along W, X_t vanishes along W. Thus there exists a tubular neighbourhood U_0 of W such that the flow $\phi_t(x)$ of X_t exists for all $x \in U_0$ and $t \in [0,1]$, and $\phi_t(U_0) \subset U$ for all $t \in [0,1]$. Now $\phi_1 : U_0 \to U_1 := \phi_1(U_0)$ is the desired diffeomorphism with $\phi_1^*\omega_1 = \omega_0$. □

Now we are ready for the main result of this section.

Proposition 3.8 (symplectic normal forms)**.** *Let ω_0, ω_1 be symplectic forms on a manifold V and $W \subset V$ a compact submanifold such that $\omega_0|_W = \omega_1|_W$. Suppose that $N := \ker(\omega_0|_W) = \ker(\omega_1|_W)$ has constant rank, and the bundles $(TW^{\omega_0}/N, \omega_0$, $(TW^{\omega_1}/N, \omega_1$ over W are isomorphic as symplectic vector bundles. Then there exist tubular neighbourhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = \mathbb{1}$ and $\phi^*\omega_1 = \omega_0$.*

Proof. By Proposition 3.3,

$$(TV|_W, \omega_0) \cong (TW/N, \omega_0) \oplus (TW^{\omega_0}/N, \omega_0) \oplus (N \oplus N^*, \omega_{\text{st}}),$$

and similarly for ω_1. By the hypotheses, the right-hand sides are isomorphic for ω_0 and ω_1. More precisely, there exists an isomorphism

$$\Psi : (TV|_W, \omega_0) \to (TV|_W, \omega_1)$$

with $\Psi|_{TW} = \mathbb{1}$. Extend Ψ to a diffeomorphism $\psi : U_0 \to U_1$ of tubular neighbourhoods such that $\psi|_W = \mathbb{1}$ and $\psi^*\omega_1 = \omega_0$ along W, and apply Lemma 3.7. □

All the normal forms are easy corollaries of this result.

Corollary 3.9 (Symplectic Neighbourhood Theorem)**.** *Let ω_0, ω_1 be symplectic forms on a manifold V and $W \subset V$ a compact submanifold such that $\omega_0|_W = \omega_1|_W$ is symplectic, and the symplectic normal bundles $(TW^{\omega_0}, \omega_0)$, $(TW^{\omega_1}, \omega_1)$*

over W are isomorphic (as symplectic vector bundles). Then there exist tubular neighbourhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = \mathbb{1}$ and $\phi^*\omega_1 = \omega_0$.

Corollary 3.10 (Isotropic Neighbourhood Theorem)**.** *Let ω_0, ω_1 be symplectic forms on a manifold V and $W \subset V$ a compact submanifold such that $\omega_0|_W = \omega_1|_W = 0$, and the symplectic normal bundles $(TW^{\omega_0}/TW, \omega_0)$, $(TW^{\omega_1}/TW, \omega_1)$ are isomorphic (as symplectic vector bundles). Then there exist tubular neighbourhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = \mathbb{1}$ and $\phi^*\omega_1 = \omega_0$.*

Corollary 3.11 (Coisotropic Neighbourhood Theorem)**.** *Let ω_0, ω_1 be symplectic forms on a manifold V and $W \subset V$ a compact submanifold such that $\omega_0|_W = \omega_1|_W$ and W is coisotropic for ω_0 and ω_1. Then there exist tubular neighbourhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = \mathbb{1}$ and $\phi^*\omega_1 = \omega_0$.*

Corollary 3.12 (Lagrangian Neighbourhood Theorem [**?**])**.** *Let $W \subset (V, \omega)$ be a compact Lagrangian submanifold of a symplectic manifold. Then there exist tubular neighbourhoods U of the zero section in T^*W and U' of W in V and a diffeomorphism $\phi : U \to U'$ such that $\phi|_W$ is the inclusion and $\phi^*\omega = \omega_{\rm st}$.*

Proof. Since W is Lagrangian, the map $v \mapsto i_v\omega$ defines an isomorphism from the normal bundle $TV/TW|_W$ to T^*W. Extend the inclusion $W \subset V$ to a diffeomorphism $\psi : U \to U'$ of tubular neighbourhoods of the zero section in T^*W and of W in V. Now apply the Coisotropic Neighbourhood Theorem to the zero section in T^*W and the symplectic forms $\omega_{\rm st}$ and $\psi^*\omega$. □

3.5 Contact manifolds and their Legendrian submanifolds

A *contact structure* ξ on a manifold M is a completely non-integrable tangent hyperplane field. According to the Frobenius condition, this means that for every nonzero local vector field $X \in \xi$ there exists a local vector field $Y \in \xi$ such that their Lie bracket satisfies $[X, Y] \notin \xi$. If α is any 1-form locally defining ξ, i.e. $\xi = \ker \alpha$, this means

$$d\alpha(X, Y) = -\frac{1}{2}\alpha([X, Y]) \neq 0.$$

So the restriction of the 2-form $d\alpha$ to ξ is nondegenerate, i.e. $(\xi, d\alpha|_\xi)$ is a symplectic vector bundle. This implies in particular that $\dim \xi$ is even and $\dim M = 2n + 1$ is odd. In terms of a local defining 1-form α, the contact condition can also be expressed as $\alpha \wedge (d\alpha)^n \neq 0$.

Remark 3.13. If $\dim M = 4k + 3$ the sign of the volume form $\alpha \wedge (d\alpha)^{2k+1}$ is independent of the sign of the defining local 1-form α, so a contact structure defines an orientation of the manifold. In particular, in these dimensions contact structures can exist only on orientable manifolds. On the other hand, a contact structure ξ on a manifold of dimension $4k+1$ is itself orientable.

Contact structures ξ in this book will always be *cooriented*, i.e., they are globally defined by a 1-form α. In this case the symplectic structure on each of the hyperplanes ξ is defined uniquely up to a positive conformal factor.

Given a J-convex hypersurface M (which is by definition cooriented) in an almost complex manifold (V,J), the field ξ of complex tangencies defines a contact structure on M which is cooriented by $J\nu$, where ν is a vector field transverse to M defining the coorientation. Conversely, any cooriented contact structure ξ arises as a field of complex tangencies on a J-convex hypersurface in an almost complex manifold: Just chose a complex multiplication J on ξ compatible with the symplectic form $d\alpha$ in the sense that $d\alpha(\cdot,J\cdot)$ is a (positive definite) inner product on ξ and extend J arbitrarily to an almost complex structure on $V:=M\times(-\epsilon,\epsilon)$.

Remark 3.14. If $\dim M=3$ then J can always be chosen integrable. However, in dimensions ≥ 5 this is not always the case, see Example ??? below.

Let $(M,\xi=\ker\alpha)$ be a contact manifold of dimension $2n+1$. An immersion $\phi:\Lambda\to M$ is called *isotropic* if it is tangent to ξ. Then at each point $x\in\Lambda$ we have $d\phi(T_x\Lambda)\subset\xi_{\phi(x)}$ and $d\alpha|_{d\phi(T_x\Lambda)}=d\big(\alpha|_{\phi(\Lambda)}\big)(x)=0$. Hence $d\phi(T_xL)$ is an isotropic subspace in the symplectic vector space $(\xi_x,d\alpha)$. In particular,

$$\dim\Lambda\leq\frac{1}{2}\dim\xi=n.$$

Isotropic immersions of the maximal dimension n are called *Legendrian*.

1-jet spaces. Let L be a manifold of dimension n. The space J^1L of 1-jets of functions on L can be canonically identified with $T^*L\times\mathbb{R}$, where T^*L is the cotangent bundle of L. A point in J^1L is a triple (q,p,z) where q is a point in L, p is a linear form on T_qL, and $z\in\mathbb{R}$ is a real number. Pick local coordinates $(q_1,\dots,q_n)$ are local coordinates on L and write covectors in T^*L as $\sum p_i dq_i$. It is easy to check that the 1-form

$$p\,dq:=\sum_{i=1}^n p_i dq_i$$

is independent of the choice of such coordinates. It is called the *canonical 1-form on* T^*L. The 2-form $dp\wedge dq:=d(p\,dq)$ is called the *canonical sympletic form on* T^*L. The 1-form $dz-p\,dq$ defines the *canonical contact structure*

$$\xi_{\mathrm{can}}:=\ker(dz-p\,dq)$$

on J^1L. A function $f:L\to\mathbb{R}$ defines a section

$$q\mapsto j^1f(q):=\Big(q,df(q),f(q)\Big)$$

of the bundle $J^1L \to L$. Since $f^*(dz - p\,dq) = df - df = 0$, this section is a Legendrian embedding in the contact manifold (J^1L, ξ). Consider the following diagram, where all arrows represent the obvious projections:

[to be added]

We call P_{Lag} the *Lagrangian projection* and P_{front} the *front projection.* Given a Legendrian submanifold $\Lambda \subset J^1L$, consider its images

$$P_{\text{Lag}}(\Lambda) \subset T^*L, \qquad P_{\text{front}}(\Lambda) \subset L \times \mathbb{R}.$$

The map $P_{\text{Lag}} : \Lambda \to T^*L$ is a Lagrangian immersion with respect to the standard symplectic structure $dp \wedge dq = d(p\,dq)$ on T^*L. Indeed, the contact hyperplanes of ξ_{can} are transverse to the z-direction which is the kernel of the projection P_{Lag}. Hence Λ is transverse to the z-direction as well and $P_{\text{Lag}}|_\Lambda$ is an immersion. It is Lagrangian because

$$P_{\text{Lag}}^* dp \wedge dq = d(p\,dq|_\Lambda) = d(dz|_\Lambda) = 0.$$

Conversely, any *exact Lagrangian immersion* $\phi : \Lambda \to T^*L$, i.e. an immersion for which the form ϕ^*, dq is exact, lifts to a Legendrian immersion $\hat{\phi} : \Lambda \to J^1L$. It is given by the formula $\hat{\phi} := (\phi, H)$, where H is a primitive of the exact 1-form $\phi^* p\,dq$ so that $\hat{\phi}^*(dz - p\,dq) = dH - \phi^* p\,dq = 0$. The lift $\hat{\phi}$ is unique up to a translation along the z-axis.

Remark 3.15. More generally, a *Liouville structure* on an even-dimensional manifold is a 1-form α such that $d\alpha$ is symplectic. For example, the form $p\,dq$ is the canonical Liouville form on the cotangent bundle T^*L. An immersion $\phi : L \to V$ into a Liouville manifold (V, α) is called *exact Lagrangian* if $\phi^*\alpha$ is exact.

Let us now turn to the front projection. The image $P_{\text{front}}(\Lambda)$ is called the *(wave) front* of the Legendrian submanifold $\Lambda \subset J^1L$. If the projection $\pi|_\Lambda : \Lambda \to L$ is nonsingular and injective, then Λ is a graph $\{\big(q, \alpha(q), f(q)\big) \mid q \in \pi(\Lambda)\}$ over $\pi(\Lambda) \subset L$. The Legendre condition implies that the 1-form α is given by $\alpha = df$. So

$$\Lambda = \{\big(q, df(q), f(q)\big) \mid q \in \pi(\Lambda)\}$$

is the graph of the 1-jet $j^1 f$ of a function $f : \pi(\Lambda) \to \mathbb{R}$. In this case the front $P_{\text{front}}(\Lambda)$ is just the graph of the function f.

In general, the front of a Legendrian submanifold $\Lambda \subset J^1L$ can be viewed as the graph of a multivalued function. Note that since the contact hyperplanes are transverse to the z-direction, the singular points of the projection $\pi|_\Lambda$ coincide with the singular points of the projection $P_{\text{front}}|_\Lambda$. Hence near each of its nonsingular points the front is indeed the graph of a function.

In general, the front can have quite complicated singularities. But when the projection $\pi|_\Lambda : \Lambda \to L$ has only "fold type" singularities, then the front itself has only "cuspidal" singularities along its singular locus as shown in Figure ??? [to be added].

Let us discuss this picture in more detail. Consider first the 1-dimensional case when $L = \mathbb{R}$. Then $J^1L = \mathbb{R}^3$ with coordinates (q, p, z) and contact structure $\ker(dz - p\,dq)$. Consider the curve in $\mathbb{R}^3$ given by the equations

$$q = 3p^2, \qquad z = 2p^3. \tag{3.4}$$

This curve is Legendrian because $dz = 6p^2dp = p\,dq$. Its front is given by (3.4) viewed as parametric equations for a curve in the (q, z)-plane. This is a semicubic parabola as shown in Figure ??? [to be added].

Generically, any singular point of a Legendrian curve in $\mathbb{R}^3$ looks like this. This means that, after a C^∞-small perturbation of the given curve to another Legendrian curve, there exists a contactomorphism (i.e. a diffeomorphism which preserves the contact structure) of a neighbourhood of the singularity which transforms the curve to the curve described by (3.4) (see [2], Chapter 1 §4). If we want to construct just C^1 Legendrian curves (and any C^1 Legendrian curve can be further C^1-approximated by C^∞ or even real analytic Legendrian curves, see Corollary 8.25), then the following characterization of the front near its cusp points will be convenient. Suppose that the two branches of the front which form the cusp are given locally by the equations $z = f(q)$ and $z = g(q)$, where the functions $f, g : [0, \epsilon) \to \mathbb{R}$ satisfy $f \le g$ (see Figure ??? [to be added]). Then the front lifts to a C^1 Legendrian curve if and only if

$$\begin{aligned} f(0) = g(0), &\qquad f'(0) = g'(0), \\ f''(q) \to -\infty \text{ as } q \to 0, &\qquad g''(q) \to +\infty \text{ as } q \to 0. \end{aligned}$$

In higher dimensions, suppose that a Legendrian submanifold $\Lambda \subset J^1L$ projects to L with only "fold type" singularities. Then along its singular locus the front consists of the graphs of two functions $f \le g$ defined on an immersed strip $S \times [0, \epsilon)$. Denoting coordinates on $S \times [0, \epsilon)$ by (s, t), the front lifts to a C^1 Legendrian submanifold if and only if

$$\begin{aligned} f(s, 0) = g(s, 0), &\qquad \frac{\partial f}{\partial t}(s, 0) = \frac{\partial g}{\partial t}(s, 0), \\ \frac{\partial^2 f}{\partial t^2}(s, t) \to -\infty \text{ as } t \to 0, &\qquad \frac{\partial^2 g}{\partial t^2}(s, t) \to +\infty \text{ as } t \to 0. \end{aligned}$$

However, in higher dimensions not all singularities are generically of fold type.

Example 3.16. Given a contact manifold $(M, \xi = \ker \alpha)$ and a Liouville manifold (V, β), their product $M \times V$ is a contact manifold with the contact form $\alpha \oplus \beta$. For example, if $M = J^1N$ and $V = T^*W$ with the canonical contact and Liouville forms, then $M \times V = J^1(N \times W)$ with the canonical contact form. A product $\Lambda \times L$ of a Legendrian submanifold $\Lambda \subset M$ and an exact Lagrangian submanifold $L \subset V$ is a Legendrian submanifold of $M \times V$. In particular, the product of a Legendrian submanifold $\Lambda \subset J^1N$ and an exact Lagrangian submanifold $L \subset T^*W$ is a Legendrian submanifold in $J^1(N \times W)$.

Local properties of contact manifolds and their isotropic submanifolds. Let $(M^{2n+1}, \xi = \ker \alpha)$ be a contact manifold and $\Lambda^k \subset M$, $0 \le k \le n$, be an isotropic submanifold. The following result is due to Darboux in the case that Λ is a point (see e.g. Appendix 4 of [1]); the extension to general Λ is straightforward and left to the reader.

Proposition 3.17 (Darboux' Theorem)**.** *Near each point on Λ there exist coordinates $(q_1, \dots, q_n, p_1, \dots, p_n, z) \in \mathbb{R}^{2n+1}$ in which $\alpha = dz - \sum p_i dq_i$ and $\Lambda = \mathbb{R}^k \times \{0\}$.*

To formulate a more global result, recall that the form $\omega = d\alpha$ defines a natural (i.e., independent of α) conformal symplectic structure on ξ. Denote the ω-orthogonal on ξ by a superscript ω. Since Λ is isotropic, $T\Lambda \subset T\Lambda^\omega$. So the normal bundle of Λ in M is given by

$$TM/T\Lambda = TM/\xi \oplus \xi/(T\Lambda)^\omega \oplus (T\Lambda)^\omega/T\Lambda \cong \mathbb{R} \oplus T^*\Lambda \oplus CSN(\Lambda).$$

Here TM/ξ is trivialized by the Reeb vector field R_α, the bundle $\xi/(T\Lambda)^\omega$ is canonically isomorphic to T^Λ via $v \mapsto i_v\omega$, and $CSN(\Lambda) := (T\Lambda)^\omega/T\Lambda$ denotes the *conformal symplectic normal bundle* which carries a natural conformal symplectic structure induced by ω. Thus $CSN(\Lambda)$ has structure group $Sp(n-k)$, which can be reduced to $U(n-k)$ by choosing a compatible complex structure.

Let (M, ξ_M) and (N, ξ_N) be two contact manifolds. A map $f : M \to N$ is called *isocontact* if $f^*\xi_N = \xi_M$, where $f^*\xi_N := \{v \in TM \mid df \cdot v \in \xi_N\}$. Equivalently, f maps any defining 1-form α_N for ξ_N to a defining 1-form $f^*\alpha_M$ for ξ_M. In particular, f must be an immersion and thus $\dim M \le \dim N$. Moreover, $df : \xi_M \to \xi_N$ is *conformally symplectic*, i.e., symplectic up to a scaling factor. We call a monomorphism $F : TM \to TN$ *isocontact* if $F^*\xi_N = \xi_M$ and $F : \xi_M \to \xi_N$ is conformally symplectic.

Proposition 3.18 (Weinstein [48])**.** *Let (M, ξ_M), (N, ξ_N) be contact manifolds with $\dim M \le \dim N$ and $\Lambda \subset M$ an isotropic submanifold. Let $f : \Lambda \to N$ be an isotropic immersion covered by an isocontact monomorphism $F : TM \to TN$. Then there exists an isocontact immersion $g : U \to N$ of a neighbourhood $U \subset M$ of Λ with $g|_\Lambda = f$ and $dg = F$ along Λ.*

Remark 3.19. (a) If ϕ is an embedding then ψ is also an embedding on a sufficiently small neighbourhood. It follows that a neighbourhood of a Legendrian submanifold Λ is contactomorphic to a neighbourhood of the zero section in the 1-jet space $J^1\Lambda$ (with its canonical contact structure).

(b) A Legendrian immersion $f : \Lambda \to (M, \xi)$ extends to a an isocontact immersion of a neighbourhood of the zero section in $J^1\Lambda$.

(c) Suppose that the conformal symplectic normal bundle of an isotropic submanifold Λ is the complexification of a real bundle $W \to \Lambda$ (i.e., the structure group of $CSN(\Lambda)$ reduces from $U(n-k)$ to $O(n-k)$). Then a neighbourhood of Λ is contactomorphic to a neighbourhood of the zero section in $J^1\Lambda \oplus (W \oplus W^*)$ (with its canonical contact structure, see Example 3.16). In this case (and only in this case) the isotropic submanifold Λ extends to a Legendrian submanifold (the total space of the bundle W).

We will need a stronger form of Weinstein theorem 3.18. Not only the contact structure, but even the contact form can be standartized near an isotropic submanifold.

Proposition 3.20. *Let λ_0, λ_1 be two contact forms for the same contact structure ξ defined on a neighborhood of an isotropic submanifold $\Lambda \subset V$. Then there exists a fixed along Λ contact isotopy $h_t : \mathcal{O}p\,(\lambda) \to \mathcal{O}p\,(\Lambda)$ such that $\lambda_1 = h_1^*\lambda_0$.*

Proof. We are following here the standard Moser homotopic method. Set $\lambda_t = (1-t)\lambda_0 + t\lambda_1,\ t \in [0,1]$. Then λ_t is a contact form for ξ for all $t \in [0,1]$. Differentiating the equation $h_t * \lambda_0 - \lambda_t$, we get, using Carna's formula for the Lie derivative:

$$i(X_t)d\lambda_t + d(\lambda_t(X_t)) = \mu, \tag{3.5}$$

where

$$X_t(h_t(x)) = \frac{dh_t(x)}{dt} \quad \text{and} \quad \mu = \lambda_1 = \lambda_0.$$

Let R_t denotes the Reeb vector field of the form λ_t, i.e. $\lambda_t(R_t) = 1$ and $i(R_t)d\lambda_t = 0$. Let us write $X_t = a_tR_t + Y_t$, where $Y_t \in \xi$ and denote $b_t := \mu(R_t)$ and $\alpha := \mu|_\xi$. Then (3.5) is equivalent to the system

$$\begin{aligned} da_t(R_t) &= b_t, \\ i(Y_t)d\lambda_t &= \alpha - da_t|_\xi. \end{aligned} \tag{3.6}$$

Let us consider a germ Σ along Λ of a hypersurface tangent to ξ along Λ. There exists a smooth function f on Σ such that $f|_\Lambda = 0$ and $df|_{\xi_\Lambda} = \alpha|_{\xi|_\Lambda}$. Note that for each t the vector field R_t is transverse to Σ on $\mathcal{O}p\,\Lambda$. Hence the first of equations (3.6) has a solution a_t on $\mathcal{O}p\,\Lambda$ which satisfies an initial condition $a_t|_\Sigma = f$. The second equation is a non-differential non-degenerate linear system of equation with respect to Y_t and hence it has a unique solution Y_t after a_t is found. Note that by our choice of f the right-hand side of the second equation vanishes along Λ, and hence $X_t|_\Lambda = (a_tR_t + Y_t)|_\Lambda = 0$. Hence the vector field X_t can be integrated to the required isotopy $h_t : \mathcal{O}p\,\Lambda \to \mathcal{O}p\,\Lambda$, fixed along Λ.

□

All the properties discussed in this section also hold for families of isotropic submanifolds. Moreover, any isotropic submanifold with boundary can be extended beyond the boundary to a slightly bigger isotropic submanifold of the same dimension.

Finally, we mention that a similar homotopy argument proves Gray's stability theorem, which states that on a closed manifold all deformations of a contact structure are diffeomorphic to the original one.

Theorem 3.21 (Gray's stability theorem [18])**.** *Let $(\xi_t)_{t\in[0,1]}$ be a smooth homotopy of contact structures on a closed manifold M. Then there exists an isotopy of diffeomorphisms $\phi_t : M \to M$ with $\phi_0 = \mathbb{1}$ and $\phi_t^*\xi_t = \xi_0$ for all $t \in [0,1]$.*

3.6 Stabilization of Legendrian submanifolds

The goal of this section is the proof of the following

Proposition 3.22. *Let $\Lambda_0 \subset (M^{2n+1}, \xi = \ker\alpha)$ be a closed orientable Legendrian submanifold and k an integer. Suppose that $n > 1$. Then there exists a Legendrian submanifold $\Lambda_1 \subset M$ and a Legendrian regular homotopy Λ_t, $t \in [0,1]$, such that the self-intersection index of the immersion $L := \cup_{t\in[0,1]}\Lambda_t \times \{t\} \subset M \times [0,1]$ equals k (mod 2 if n is even).*

A local construction. The proof of Proposition 3.22 is base on a *stabilization* procedure which we will now describe. Consider the front projection of a (not necessarily closed) orientable Legendrian submanifold $\Lambda_0 \subset \mathbb{R}^{2n+1}$. Suppose that $P_{\text{front}}(\Lambda_0)$ intersects $B^n \times [-1,2]$ in the two oppositely oriented branches $\{z = 0\}$ and $\{z = 1\}$. Let $f : B^n \to (-1,2)$ be a function which equals zero near ∂B^n and has no critical points on level 1. Replacing the branch $\{z = 0\}$ over B^n by $\{z = tf(q)\}$ we obtain a family of Legendrian immersions $\Lambda_t \subset \mathbb{R}^{2n+1}$, $t \in [0,1]$. Note that the set $\{q \in B^n \mid f(q) \geq 1\}$ is a smooth n-manifold with boundary. Denote by $\chi(\{f \geq 1\})$ its Euler characteristic.

Lemma 3.23. *The self-intersection index of the immersion $L := \cup_{t\in[0,1]}\Lambda_t \times \{t\} \subset M \times [0,1]$ equals*

$$I_L = (-1)^{n(n-1)/2}\chi(\{f \geq 1\})$$

(mod 2 if n is even).

Proof. Perturb f such that all critical points above level 1 are nondegenerate and lie on distinct levels. Self-intersections of L occur precisely when $t_0 f$ has a critical point q_0 on level 1 for some $t_0 \in (0,1)$. By the Morse Lemma, we find coordinates near q_0 in which $q_0 = 0$ and f has the form

$$f(q) = a_0 - \frac{1}{2}\sum_{i=1}^{k} q_i^2 + \frac{1}{2}\sum_{i=k+1}^{n} q_i^2,$$

where $a_0 = f(q_0) = 1/t_0$ and k is the Morse index of q_0. The p-coordinates on the branch $\{z = tf(q)\}$ of Λ_t near q_0 are given by

$$p_i = \frac{\partial(tf)}{\partial q_i} = \begin{cases} -tq_i & i \leq k, \\ +tq_i & i \geq k+1. \end{cases}$$

Thus the tangent spaces in $T(\mathbb{R}^{2n+1} \times [0,1]) = \mathbb{R}^{2n+2}$ of the two intersecting branches of L corresponding to $\{z = 1\}$ and $\{z = t_0 f(q)\}$ are given by

$$T_1 = \{p_1 = \cdots = p_n = 0, z = 0\},$$
$$T_2 = \{p_i = -t_0 q_i \text{ for } i \leq k, p_i = +t_0 q_i \text{ for } i \geq k+1, z = a_0 t\}.$$

Without loss of generality (because the self-intersection index does not depend on the orientation of L) suppose that the basis $(\partial_{q_1}, \dots, \partial_{q_n}, \partial_t)$ represents the orientation of T_1. Since the two branches of Λ_0 are oppositely oriented, the orientation of T_2 is then represented by the basis

$$\big(\partial_{q_1} - t_0\partial_{p_1}, \dots, \partial_{q_n} + t_0\partial_{p_n}, -(\partial_t + a_0\partial_z)\big).$$

Hence the orientation of (T_1, T_2) is represented by

$$(\partial_{q_1}, \dots, \partial_{q_n}, \partial_t, -\partial_{p_1}, \dots, -\partial_{p_n}, -\partial_z),$$

which equals $(-1)^{k+n+n(n-1)/2}$ times the complex orientation

$$(\partial_{q_1}, \partial_{p_1}, \dots, \partial_{q_n}, \partial_{p_n}, \partial_z, \partial_t)$$

of $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$. So the local intersection index of L at a critical point q equals

$$I_L(q) = (-1)^{\mathrm{ind}_f(q)+n+n(n-1)/2}$$

(mod 2 if n is even), where $\mathrm{ind}_f(q)$ is the Morse index of q.

On the other hand, for a vector field v on a compact manifold N with boundary which is outward pointing along the boundary and has only nondegenerate zeroes we have *Poincaré-Hopf Index Theorem* holds: The sum of the indices of v at all its zeroes equals the Euler characteristic of M (see [24]). Note that if v is the gradient vector field of a Morse function f, then the index of v at a critical point q of f equals $(-1)^{\mathrm{ind}_f(q)}$. Applying the Poincaré-Hopf Index Theorem to the gradient of the Morse function $-f$ on the manifold $\{f \geq 1\} = \{-f \leq -1\}$ (which is outward pointing along the boundary because f has no critical point on level 1), we obtain

$$\begin{aligned}\chi(\{f \geq 1\} &= \sum_q \mathrm{ind}_{\nabla(-f)}(q) = \sum_q (-1)^{\mathrm{ind}_{-f}(q)} = \sum_q (-1)^{n-\mathrm{ind}_f(q)} \\ &= (-1)^{n(n-1)/2} \sum_q I_L(q) = (-1)^{n(n-1)/2} I_L.\end{aligned}$$

□

Proof of Proposition 3.22. Since all Legendrian submanifolds are locally isomorphic, a neighborhood in M of a point on Λ_0 is contactomorphic to a neighbourhood in $\mathbb{R}^{2n+1}$ of a point on a standard cusp $3z^2 = 2q_1^2$. Thus the front consists of two branches $\{z = \pm\sqrt{\frac{2}{3}q_1^3}\}$ joined along the singular locus $\{z = q_1 = 0\}$. Now deform the branches to $\{z = \pm\varepsilon\}$ over a small ball disjoint from the singular locus, thus (after rescaling) creating two parallel branches over a ball as in Lemma **??**. Now deform Λ_0 to Λ_1 as in Lemma **??**, for some function $f : B^n \to (-1, 2)$. Then Proposition **??** follows from Lemma **??**, provided that we arrange $\chi(\{f \geq 1\}) = k$ for a given integer k if $n > 1$.

Thus it only remains to find for $n > 1$ an n-dimensional submanifold-with-boundary $N \subset \mathbb{R}^n$ of prescribed Euler characteristic $\chi(N) = k$ (then write $N = \{f \geq 1\}$ for a function $f : N \to [1, 2)$ without critical points on the boundary). Let N_+ be a ball in $\mathbb{R}^n$, thus $\chi(N_+) = +1$. Let N_- be a smooth tubular neighbourhood in $\mathbb{R}^n$ of a figure eight in $\mathbb{R}^2$, thus $\chi(N_-) = -1$ (here we use $n \geq 2$!). So we can arrange $\chi(N)$ to be any integer by taking disjoint unions of copies of $N_\pm$. □

Remark 3.24. The preceding proof fails for $n = 1$ because a 1-dimensional manifold with boundary always has Euler characteristic $\xi \geq 0$. Therefore for $n = 1$ the local construction in Lemma 3.23 allows us only to realize *positive* values of the self-intersection index I_L. As explained in Appendix **??**, this failure to create negative I_L is unavoidable in view of Bennequin's inequality. However, no analog of Bennequin's inequality exists in *overtwisted* contact 3-manifolds, and we will show in Section 4.6 how to realize any value of the self-intersection index in that case.

Chapter 4

The h-principles

We begin by reviewing necessary facts about smooth immersions and embeddings.

4.1 Immersions and embeddings

For a closed subset $A \subset X$ of a topological space, we denote by $\mathcal{O}p\, A$ a sufficiently small (*but not specified*) open neighbourhood of A.

The h-principle for immersions. Let M, N be manifolds. A *monomorphism* $F : TM \to TN$ is a fibrewise injective bundle homomorphism. Any immersion $f : M \to N$ gives rise to a monomorphism $df : TM \to TN$. We denote by $\mathrm{Mon}(TM, TN)$ the space fibrewise injective bundle homomorphisms, and by $\mathrm{Imm}(M, N)$ the space of immersions. Given a closed subset $A \subset M$ an immersion $h : \mathcal{O}p\, A \to N$ we denote by $\mathrm{Imm}(M, N; A, h)$ the subspace of $\mathrm{Imm}(M, N)$ which consists of immersions equal to h on $\mathcal{O}p\, A$. Similarly, the notation $\mathrm{Mon}(TM, TN; A, h)$ stands for the subspace of $\mathrm{Mon}(TM, TN$ of homomorphisms which coincide with dh on $\mathcal{O}p\, A$. Extending S. Smale's theory of immersions of spheres (see [**?**, **?**]) M.Hirsch proved the following h-principle (see also [23],[12]):

Theorem 4.1 (Hirsch [28])**.** *For* $\dim M < \dim N$*, the map* $f \mapsto df$ *defines a homotopy equivalence between the spaces* $\mathrm{Imm}(M, N; A, h)$ *and* $\mathrm{Mon}(TM, TN; A, h)$ *for any immersion* $h \in \mathcal{O}p\, A$*. In particular, any monomorphism* $F \in \mathrm{Mon}(TM, TN; A, h)$ *is homotopic to the differential* df *of an immersion* $f : M \to N$ *which coincides with* h *on* $\mathcal{O}p\, A$*. Given a homotopy* $F_t \in \mathrm{Mon}(TM, TN)$*,* $t \in [0, 1]$*, between the differentials* $F_0 = df_0$ *and* $F_1 = df_1$ *of two immersions* $f_0, f_1 \in \mathrm{Imm}(M, N; A, h)$*, one can find a regular homotopy* $f_t \in \mathrm{Imm}(M, N; A, h)$*,* $t \in [0, 1]$*, such that the paths* F_t *and* df_t*,* $t \in [0, 1]$*, are homotopic with fixed ends.*

For example, if M is parallelizable, i.e. $TM \cong M \times \mathbb{R}^k$, the inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$

gives rise to a monomorphism $TM = M \times \mathbb{R}^k \to T(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$, $(x, v) \mapsto (0, v)$. Thus Hirsch's theorem implies that every parallelizable closed manifold M^k can be immersed into $\mathbb{R}^{k+1}$.

Immersions of half dimension. Next we describe results of Whitney [50] on immersions of half dimension. Fix a closed connected manifold M^n of dimension $n \geq 2$ and an oriented manifold N^{2n} of double dimension. Let $f : M \to N$ be an immersion whose only self-intersections are transverse double points. Then if M is orientable and n is even we assign to every double point $z = f(p) = f(q)$ an integer $I_f(z)$ as follows. Choose an orientation of M. Set $I_f(z) := \pm 1$ according to whether the orientations of $df(T_pM)$ and $df(T_qM)$ together determine the orientation of N or not. Note that this definition depends neither on the order of p and q (because n if even), nor on the orientation of M. Define the *self-intersection index*

$$I_f := \sum_z I_f(z) \in \mathbb{Z}$$

as the sum over all self-intersection points z. If n is odd or M non-orientable define $I_f \in \mathbb{Z}_2$ as the number of self-intersection points modulo 2.

Theorem 4.2 (Whitney [50])**.** *For a closed connected manifold M^n and an oriented manifold N^{2n}, $n \geq 2$, the following holds.*

(a) The self-intersection index is invariant under regular homotopies.

(b) The self-intersection index of a totally regular immersion $f : M \to N$ can be changed to any given value by a local modification (which is of course not a regular homotopy).

(c) If $n \geq 3$, any immersion $f : M \to N$ is regularly homotopic to an immersion with precisely $|I_f|$ transverse double points (where $|I_f|$ means 0 resp. 1 for $I_f \in \mathbb{Z}_2$).

Since every immersion of half dimension is regularly homotopic to an immersion with transverse self-intersections ([49], see also [29]), Part (a) allows to define the self-intersection index for every immersion $f : M \to N$. Since every n-manifold immerses into $\mathbb{R}^{2n}$, Parts (b) and (c) imply (the cases $n = 1, 2$ are treated by hand)

Corollary 4.3 (Whitney Embedding Theorem [50])**.** *Every closed n-manifold M^n, $n \geq 1$, can be embedded in $\mathbb{R}^{2n}$.*

Remark 4.4. The preceding results continue to hold if M has boundary, provided that for immersions and during regular homotopies no self-intersections occur on the boundary.

Remark 4.5. For $n = 1$ Whitney [50] defines a self-intersection index $I_f \in \mathbb{Z}$. With this definition, all the preceding results continue to hold for $n = 1$ (note e.g. that $\pi_1 V_{2,1} = \mathbb{Z}$).

Isotopies. Finally, we discuss *isotopies*, i.e. homotopies through embeddings. Consider a closed connected orientable k-manifold M^k and an oriented $(2k+1)$-manifold N^{2k+1}. Let $f_t : M \to N$ be a regular homotopy between embeddings

$f_0, f_1 : M \hookrightarrow N$. Define the immersion of half dimension $F : M \times [0,1] \to N \times [0,1]$, $F(x,t) := \big(f_t(x), t\big)$. Its self-intersection index $I_{\{f_t\}} := I_F$ is an invariant of f_t in the class of regular homotopies with fixed endpoints f_0, f_1. Recall that $I_{\{f_t\}}$ takes values in $\mathbb{Z}$ if k is odd and $\mathbb{Z}_2$ if k is even.

Theorem 4.6 (Whitney)**.** *If $k > 1$ and N is simply connected, then f_t can be deformed through regular homotopies with fixed endpoints to an isotopy if and only if $I_{\{f_t\}} = 0$.*

The proof uses the following

Lemma 4.7. *Let M, N, Λ be manifolds and $F : \Lambda \times M \to N$ a smooth map. If $2\dim M + \dim\Lambda < \dim N$, then F can be C^∞-approximated by a map $\tilde F$ such that $\tilde F(\lambda,\cdot)$ is an embedding for all $\lambda \in \Lambda$. Moreover, if F is already an embedding on a compact subset $K \subset \Lambda \times M$ we can choose $\tilde F = F$ on K.*

The case $\Lambda = [0,1]$ is due to Whitney [49].

Proof of Theorem 4.6. The argument is an adjustment of the Whitney trick [50]. Take two self-intersection points $Y_0 = (y_0, t_0), Y_1 = (y_1, t_1) \in N \times (0,1)$ of the immersion $F : M^k \times [0,1] \to N^{2k+1} \times [0,1]$ defined above. If $k+1$ is even we assume that the intersection indices of these points have opposite signs. Each of the double points y_0, y_1 is the image of two distinct points $x_0^\pm, x_1^\pm \in M$, i.e. we have $f_{t_0}(x_0^\pm) = y_0$ and $f_{t_1}(x_1^\pm) = y_1$. As $k > 1$, we find two embedded paths $\gamma^\pm : [t_0, t_1] \to M$ such that $\gamma^\pm(t_0) = x_0^\pm$, $\gamma^\pm(t_1) = x_1^\pm$, and $\gamma^+(t) \neq \gamma^-(t)$ for all $t \in [t_0, t_1]$. We claim that there exists a smooth family of paths $\delta_t : [-1,1] \to M$, $t \in [t_0, t_1]$, such that

- $\delta_t(\pm 1) = \gamma^\pm(t)$ for all $t \in [t_0, t_1]$;
- $\delta_{t_0}(s) = y_0$, $\delta_{t_1}(s) = y_1$ for all $s \in [-1,1]$;
- δ_t is an embedding for all $t \in (t_0, t_1)$.

Indeed, a family with the first two properties exists because N is simply connected. Moreover, we can arrange that δ_t is an embedding for $t \neq t_0, t_1$ close to t_0, t_1. Now we can achieve the third property by Lemma 4.7 because $2 \cdot 1 + 1 < 2k + 1$. Define

$$\Delta : [t_0, t_1] \times [-1,1] \to N \times [0,1], \qquad (t,s) \mapsto \big(\delta_t(s), t\big).$$

Then Δ is an embedding on $(t_0, t_1) \times [-1,1]$ and $\Delta(t_0 \times [-1,1]) = Y_0$, $\Delta(t_1 \times [-1,1]) = Y_1$. Thus Δ serves as a Whitney disk for elimination of the double points Y_0, Y_1 of the immersion F. Due to the special form of Δ, Whitney's elimination construction ([50], see also [37]) can be performed in such a way that the modified immersion $\tilde F$ has the form $\tilde F(x,t) := \big(\tilde f_t(x), t\big)$ for a regular homotopy $\tilde f_t : M \to N$ such that the paths $f_t, \tilde f_t \in \mathrm{Imm}(M,N)$, $t \in [0,1]$, are homotopic. Hence the repeated elimination of pairs of opposite index intersection points of the immersion F results in the required isotopy between f_0 and f_1. □

4.2 The h-principle for isotropic immersions

The following h-principle was proved by Gromov in 1986 ([23], see also [12]).

Let $(M.\xi)$ be a contact manifold of dimension $2n+1$ and J a compatible almost complex structure on ξ. Let Λ be a manifold of dimension $k \leq n$, and $A \subset \Lambda$ its closed submanifold. Let $h : \mathcal{O}p\, A \to M$ be an isotropic immersion. We denote by $\mathrm{Iso}(\Lambda, M; A, h)$ the space of isotropic immersions $F \to M$ which coincide with h on $\mathcal{O}p\, A$, and by $\mathrm{Real}(T\Lambda, \xi; A, dh)$ the space of injective totally real homomorphisms $T\Lambda \to \xi$ which coincide with dh on $\mathcal{O}p\, A$. The map $f \mapsto df$, $f \in \mathrm{Iso}(\Lambda, M; A, h)$ provides an inclusion $d : \mathrm{Iso}(F, M; A, dh) \hookrightarrow \mathrm{Real}(T\Lambda, \xi)$.

Theorem 4.8 (Gromov's h-principle for isotropic immersions; contact case, see [23] and also [12])**.** *The map $d : \mathrm{Iso}(\Lambda, M; A, h) \hookrightarrow \mathrm{Real}(T\Lambda, \xi; A, dh)$ is a homotopy equivalence. In particular, given $\Phi \in \mathrm{Real}(T\Lambda, \xi, A, dh)$ one can find $f \in \mathrm{Iso}(\Lambda, M; A, h)$ such that df and Φ are homotopic in $\mathrm{Real}(T\Lambda, \xi; A, dh)$. Moreover, f can be chosen C^0-close to the map $\phi : \Lambda \to M$ covered by the homomorphism Φ. Given two isotropic immersions $f_0, f_1 : F \to M$ from $\mathrm{Iso}(\Lambda, M; A, h)$ and a homotopy $\Phi_t \in \mathrm{Real}(T\Lambda, \xi; A, dh)$, $t \in [0,1]$, connecting df_1 and df_1 one can find a regular homotopy $f_t \in \mathrm{Iso}(F, M; h)$ connecting df_0 and df_1 such that the paths Φ_t and df_t, $t \in [0,1]$, are homotopic in $\mathrm{Real}(T\Lambda, \xi; A, dh)$ with fixed end-points. Moreover, f_t can be chosen C^0-close to the family $\phi_t : \Lambda \to M$ covered by the homotopy Φ_t.*

Combining the preceding theorem with Hirsch's Immersion Theorem 4.1 yields

Corollary 4.9. *Let Λ, M, A, h be as in Theorem 4.8. Suppose that $f_0 : \Lambda \to M$ is an immersion which coinsides with an isotropic immersion on $\mathcal{O}p\, A$, F_t is a family of monomorphisms $TF \to TN$ such that $F_0 = df_0$, $F_t = dh$ on $\mathcal{O}p\, A$ for all $t \in [0,1]$, and $F_1 \in \mathrm{Real}(T\Lambda, M; A, dh)$. Then there exists a regular homotopy $f_t : F \to M$ such that*

- *$f_1 \in \mathrm{Iso}(\Lambda, M; A, h)$;*
- *$f_t = h$ on $\mathcal{O}p\, A$, $t \in [0,1]$;*
- *there exists a homotopy F_t^s, $s \in [0,1]$, of paths in $\mathrm{Mon}\big(T\Lambda, TM; A, dh\big)$ such that $F_t^0 = df_t$ and $F_t^1 = F_t$ for all $t \in [0,1]$, $F_0^s = df_0$, $F_t^s = df_0$ on $\mathcal{O}p\, A$ for all $s, t \in [0,1]$, and $F_1^s \in \mathrm{Real}(TF, \xi; A, dh)$ for all $s \in [0,1]$.*

For later use, let us reformulate the homotopy conditions in Theorem 4.8. Fix compatible complex structures J_M, J_N on ξ_M, ξ_N and positive transversal vector fields v_M, v_N. Since $Sp(2n)$ and $Gl(n, \mathbb{C})$ both deformation retract onto $U(n)$, the space of isocontact monomorphisms $TM \to TN$ is homotopy equivalent to the space of monomorphisms $F : TM \to TN$ for which $F(v_M) = v_N$ and $F : (\xi_M, J_M) \to (\xi_N, J_N)$ is complex linear. Since the spaces of compatible complex structures and positive transverse vector fields are contractible, this homotopy equivalence does not depend on the choice of J_M, J_N, v_M, v_N.

Here is yet another reformulation. Extend J_M to an almost complex structure on $\mathbb{R}\times M$ such that $\eta_M := -J_M v_M$ has positive $\mathbb{R}$-component, and similarly for J_N. Then any monomorphism $F: TM \to TN$ with $F(v_M) = v_N$ and $F|_\xi : \xi_M \to \xi_N$ complex linear extends canonically to a complex linear monomorphism $F^{\rm st}: T(\mathbb{R}\times M) \to T(\mathbb{R}\times N)$ via $F^{\rm st}(\eta_M) := \eta_N$. Conversely, if $\dim M < \dim N$ or the manifold M is open, then any complex monomorphism $G: T(\mathbb{R}\times M) \to T(\mathbb{R}\times N)$ is homotopic in the space of complex isomorphisms to a stabilization $F^{\rm st}$ of a monomorphism $F: TM \to TN$. Indeed, this amounts to finding a non-vanishing homotopy between the two sections $G(\eta_M)$ and η_N of the $(\dim N+1)$-dimensional bundle $g^*T(\mathbb{R}\times N) \to M$, where $g: M \to N$ is the map underlying G. This is always possible if $\dim M < \dim N$ or M is open because the only obstruction, the relative Euler class, lives in $H^{\dim N+1}(M\times[0,1], M\times\{0,1\}) = 0$.

4.3 The h-principle for isotropic embeddings

We will use the following general position observation

Lemma 4.10. *Let $\dim \Lambda = k = n - q$, $q \geq 0$, Then any q-dimensional family of isotropic immersions $\Lambda \to (M,\xi)$ can be C^∞-approximated by a family of isotropic embeddings.*

In particular, if $k < n$ then the word "immersion" in Corollary 4.9 can be replaced by "embedding".

It turns out that of $n > 1$ then, using the stabilization trick from Section 3.6 and Whitney's Theorem 4.6, this can be done even for $k = n$, i.e. one can prove the following h-principle for isotropic *embeddings* rather than immersions. For $n = 1$ the analogous claim is false.

Proposition 4.11. *Let (M^{2n+1},ξ), $n > 1$, be a contact manifold with compatible almost complex structure J on $\mathbb{R}\times M$. Let Λ^k, $k \leq n$, be a closed manifold. Let $f_0 : \Lambda \hookrightarrow M$ be an embedding and $F_t : T(\mathbb{R}\times J^1\Lambda)|_\Lambda \to T(\mathbb{R}\times M)$ be a homotopy of real monomorphisms such that $F_0 = 1\!\!1 \times df_0|_\Lambda$ and F_1 is complex linear. Then there exists an* isotopy *of embeddings $f_t : \mathcal{O}p\,\Lambda \to M$ on an open neighbourhood $\mathcal{O}p\,\Lambda \subset J^1\Lambda$ of the zero section such that f_1 is an isocontact embedding, and there exists a homotopy F_t^s, $s \in [0,1]$ of paths in $\mathrm{Mon}\big(T(\mathbb{R}\times \mathcal{O}p\,\Lambda), T(\mathbb{R}\times M)\big)$ such that $F_t^0 = 1\!\!1 \times df_t|_\Lambda$ and $F_t^1 = F_t$ for all $t \in [0,1]$, $F_0^s = 1\!\!1 \times df_0|_\Lambda$ and F_1^s is complex linear for all $s \in [0,1]$. Moreover, we can arrange that $f_t(\Lambda)$ is C^0-close to $f_0(\Lambda)$ for all $t \in [0,1]$.*

Proof. By applying Corollary 4.9 we can satisfy all the conditions of the theorem, except that f_1 will be an immersion rather than an embedding and f_t will be a regular homotopy rather than an isotopy. Of course, it is enough to arrange for the restriction $f_t|_\Lambda$ to be an isotopy. We will keep the notation f_t for this restriction.

By Lemma 4.10, after a C^∞-small isotropic regular homotopy, we may assume that f_1 is an isotropic embedding.

In the subcritical case $k < n$, a generic perturbation of f_t, fixing f_0 and f_1, will turn f_t into a smooth isotopy (Lemma 4.7).

Consider now the Legendrian case $k = n$. We will deform the regular homotopy f_t to an isotopy, keeping the end f_0 fixed and changing f_1 via a Legendrian isotopy. According to Whitney's Theorem 4.6, in order to deform the path f_t to an isotopy keeping *both* ends fixed we need the equality $I_{\{f_t\}} = 0$. On the other hand, according to Proposition 3.22, if $n > 1$ then for any Legendrian embedding g_0 there exists a Legendrian regular homotopy g_t with any prescribed value of the Whitney invariant $I_{\{g_t\}}$. Hence combining f_t, $t \in [0, 1]$, with an appropriate Legendrian regular homotopy f_t, $t \in [1, 2]$, we obtain a regular homotopy f_t, $t \in [0, 2]$, with

$$I_{\{f_t\}_{t\in[0,2]}} = 0.$$

By Whitney's Theorem 4.6, $\{f_t\}$ can be further deformed, keeping the ends f_0 and f_2 fixed, to the required isotopy. □

4.4 The h-principle for totally real embeddings

Proposition 4.12. *[see [23], [12]] Let (V, J) be an almost complex manifold of dimension $2n$, and $f : F \to V$ a smooth real embedding of a k-dimensional manifold F. Suppose that there exists a homotopy Φ_t, $t \in [0, 1]$, of $\Phi_0 = df$ into a totally real homomorphism $\Phi_1 = \Phi : TF \to TV$. Then there exists a C^0-small isotopy of f to a totally real embedding $g : F \to V$. If the embedding f is totally real on a neighborhood $\mathcal{O}p\, A$ of a closed subset $A \subset F$, and the homotopy Φ_t is fixed on $\mathcal{O}p\, A$ then the isotopy f_t can also be chosen fixed on $\mathcal{O}p\, A$.*

4.5 Discs attached to J-convex boundary

Proposition 4.14 below, which is a combination of h-principles discussed in this chapter, will play an important role in proving the main results of this book.

Let (V, J) be an almost complex manifold and $W \subset V$ a domain with smooth boundary ∂W. Given a k-disc $D \subset V \setminus \operatorname{Int} W$ with $D \cap \partial W = \partial D$ and which transversely intersects ∂W, we say that D is transversely attached to W in V. We say that D is J-*orthogonally* attached to W if $J(TD|_{\partial D}) \subset T(\partial W)$. Note that this implies that ∂D is tangent to the distribution $\xi = T(\partial W) \cap JT(\partial W)$. In particular, if ∂W is J-convex then ∂D is an isotropic submanifold for the contact structure ξ.

Remark 4.13. Note that any totally real manifold transversely attached to ∂W along an isotropic submanifold is isotopic relative its boundary to a J-orthogonal one through a totally real isotopy.

Proposition 4.14. *Suppose that (V, J) is an almost complex manifold of dimension $2n$, $W \subset V$ a domain with smooth J-convex boundary and D is a k-disc, $k \leq n$, transversely attached to W in V. Then there exists a C^0-small isotopy of D through transversally attached discs to a totally real disc D', which is J-orthogonal to ∂W.*

Proof. Let us denote by f the inclusion $D \hookrightarrow V$. There exists a homotopy $\Phi_t : TD \to TV$, $t \in [0,1]$, such that differential $\Phi_0 = df$, Φ_1 is totally real, and Φ_t is injective for all $t \in [0,1]$. We can assume, without a loss of generality that

a) $\Phi_1(T\partial D) \subset \xi$ and

b) $\Phi_t(T\partial D) \subset T\partial W$

for all $t \in [0,1]$. Indeed, we have

$$\pi_{k-1}(V^{\mathbb{C}}_{n,k-1}, V^{\mathbb{C}}_{n-1,k-1}) = 0, \quad \text{and}$$
$$\pi_{k-1}(V^{\mathbb{R}}_{2n,k-1}, V^{\mathbb{R}}_{2n-1,k-1}) = 0,$$

for $k \leq n$, where we denote by $V^{\mathbb{C}}$ and $V^{\mathbb{R}}$ complex and real Stieffel manifolds, respectively. The first equation implies a) and the second one b).

The restriction $\Phi_t|_{T(\partial D)}$ gives us a homotopy $\widetilde{\Phi}_t : T(\partial D) \to T(\partial W)$. Next, we use Proposition 4.11 to construct an isotopy $g_t : \partial D \to \partial W$ such that

(i) $g_0 = f|_{\partial D}$, g_1 is isotropic and

(ii) the path of homomorphisms $df_t : T(\partial D) \to T(\partial W)$, $t \in [0,1]$ is homotopic to $\widetilde{\Phi}_t$ in the class of paths of injective homomorphisms beginning at f and ending at a totally real homomorphism $T(\partial D) \to \xi$.

Let us extend the isotopy g_t to an isotopy $f_t : D \to V \setminus \operatorname{Int} W$ of $f_0 = f$ through discs transversely attached to W. According to Remark 4.13 we can assume that the disc $f_1(D)$ is J-orthogonal to ∂W. We claim that there exists a homotopy $\Psi_t : TD \to TV$, $t \in [0,1]$, through injective homomorphisms such that

a) $\Psi_0 = df_1 : TD \to TV$;

b) Ψ_1 is a totally real homomorphism, and

c) $\Psi_t = df_1$ on $TD|_{\partial D}$.

Indeed, consider first a homotopy

$$\Psi_t = \begin{cases} df_{1-2t}, & t \in [0, \frac{1}{2}]; \\ \Phi_{2t-1}, & t \in (\frac{1}{2}, 1]. \end{cases}$$

The homotopy Ψ_t satisfies the above conditions a) and b), but not c). However, by construction the path $\Psi_t|_{TD|_{\partial D}}$ is homotopic through paths with fixed ends to a path of totally real (injective) homomorphisms. Hence, the homotopy Φ_t can be modified to satisfy the condition c) as well.

It remains to apply Gromov's h-principle for totally real embeddings 4.12 in order to construct a fixed along ∂D, together with its differential, isotopy $f_t : D \to V \setminus \operatorname{Int} W$, $t \in [1,2]$which ends at a totally real disc $f_2 : D \to V \setminus \operatorname{Int} W$ which is J-orthogonal to ∂W. Finally note that all the isotopies provided by Propositions 4.11 and 4.12 can be chosen C^0-small. □

4.6 The three-dimensional case

[to be added]

Chapter 5

Weinstein structures

5.1 Convex symplectic manifolds

We review in this section some notions introduced in [**?**].

Let (V, ω) be an exact symplectic manifold. A primitive α, $d\alpha = \omega$, is called a *Liouville form* on V. The vector field X, ω-dual to α, i.e. such that $i(X)\omega = \alpha$ is called *Liouville vector field*. The equation $d\alpha = \omega$ is equivalent to $L_X\omega = \omega$. If X integrates to a flow $X^t : V \to V$. Then $(X^t)^*\omega = e^t\omega$, i.e. the Liouville field X is symplectically *expanding*, while $-X$ is *contracting*. By a *Liouville manifold* we will mean a pair (V, X) where X is an expanding vector field.

A symplectic manifold (V, ω) is called *symplectically convex* if it admits a *complete* expanding vector field X and an exhaustion $V = \bigcup\limits_{k=1}^{\infty} V^k$ by compact domains $V^k \subset V$ with boundaries transversal to X which are invariant under the contracting flow $X^{-t}, t > 0$.[1]

The set $\mathrm{Core(V)} = \bigcup\limits_{\mathrm{k=1}}^{\infty} \bigcap\limits_{\mathrm{t>0}} \mathrm{X^{-t}(V^i)}$ is independent of the choice of the exhausting sequence of compact sets V_k and is called the *core* of the convex manifold (V, ω, X). Note that

Lemma 5.1. $\operatorname{Int}\operatorname{Core} = \varnothing$.

Proof. For each compact set V_k we have

$$\mathrm{Volume(X^{-t}(V_k)) = e^{-t}\frac{1}{n!}\int\limits_{V_k} \omega^n \underset{t\to\infty}{\longrightarrow} = 0}\,,$$

[1]This notion of symplectic convexity is slightly more restrictive than one given in [10]. However, the authors do not know if there examples of symplectic manifolds convex in one sense but not the other.

and hence $\mathrm{Volume}(\bigcap_{t>0} X^{-t}(V_i))0$. □

We say that a convex symplectic manifold (V, ω) has a *cylindrical end* if it admits X which is transverse to a hypersurface $\Sigma \subset V$ which bounds a compact domain Ω, and such that X has no zeros outside of Ω. In this case, $V \setminus \mathrm{Int}\,\Omega$, splits as $\Sigma \times [0, \infty)$ and the 1-form $\alpha = i(X)\omega$ can be written as $e^t(\lambda)$, where the coordinate $t \in \mathbb{R}$ is the parameter of the flow and $\lambda = \alpha|_\Sigma$. The form λ is contact, and thus $V \setminus \mathrm{Int}\,\Omega$ can be identified with the positive half of the symplectization of the contact manifold $(V, \xi = \{\lambda = 0\})$. In fact, the whole symplectization of (V, ξ) sits in V as $\bigcup_{t \in \mathbb{R}} X^t(\Sigma)$ and this embedding is canonical in the sense that the image is independent of the choice of Σ. The complement $\mathrm{Core(V)} = \mathrm{V} \setminus \bigcup_{t\in\mathbb{R}} \mathrm{X}^t(\Sigma)$ is called the *core* of V. A choice of X define the contact manifold (Σ, ξ) canonically. We will write $(\Sigma, \xi) = \partial(V, X)$ and call it the *ideal contact boundary* of the Liouville manifold (X, ω) with a cylindrical end.

We do not know whether the ideal boundary depends on the choice of the Liouville field X which satisfies the cylindrical end property. The answer depends on the following open problem:

Problem 5.1. Does symplectomorphism of symplectizations imply contactomorphism of contact manifolds?

Note that all known invariants of contact manifolds (e.g contact homology and other SFT-invariants) depend only on their symplectizations, and hence cannot distinguish contact manifolds with the same symplectization.

Contact manifolds which arise as ideal boundaries of Liouville symplectic manifolds with cylindrical end are called *strongly symplectically fillable.*

5.2 Deformations of convex sympletic structures

A homotopy (V, ω_s, X_s), $s \in [0, 1]$, of convex Liouville manifolds is called an*elementary homotopy of compact type* there exists a smooth family of exhaustions $V = \bigcup_{k=1}^{\infty} V_s^i$ by compact domains $V_s^k \subset V$ with smooth boundaries transversal to X_s which are invariant under the contracting flow $X_s^{-t}, t > 0, s \in [0, 1]$. A homotopy (V, ω_s, X_s), $s \in [0, 1]$ is called of *compact type*, if it is a composition of finitely many elementary homotopies of contact type.

Proposition 5.2. *Let (V, ω_t, X_s), $s \in [0, 1]$ be a contact type homotopy of convex Liouville manifolds. Then there exists a diffeotopy $h_s : V \to V$ such that $\omega_s = h_s^* \omega_0$ for all $s \in [0, 1]$.*

Proof. Denote by Σ_s^i the boundary ∂V_s^i, by α_s the Liouville form dual to X_s, and by ξ_s^i the contact structure induced on Σ_s^i by the contact form $\alpha_s|_{\Sigma_s^i}$,

$s \in [0,1], i = 1, \dots$. By Gray's stability theorem there are families of contactomorphisms

$$\psi^i_s : (\Sigma^i_0, \xi^i_0) \to (\Sigma^i_s, \xi^i_s),$$

so that $(\psi^i_s)^* \alpha^i_s = e^{f^i_s} \alpha^i_0$. Then the map $\Psi^s_i = X_s^{-f^s_i} \circ \psi^s_i : \Sigma^0_i \to V$ satisfies $(\Psi^i_s)^* \alpha_s = \alpha^i_0)$ and hence canonically extends to a map still denoted by $\Psi^i_s : \mathcal{O}p\, \Sigma^i_0 \to \mathcal{O}p\,(X_s^{-f^i_s} \Sigma^i_s)$, which maps trajectories of X_0 to trajectories of X_s and satisfies $(\Psi^i_s)^* \alpha_s = \alpha_0$. Let us extend anyhow the symplectomorphism thus defined on $\mathcal{O}p \left(\bigcup\limits_1^\infty \Sigma^i_0 \right)$ as a diffeomorphism Ψ_s on the rest of V. Then compactness of each of the domains $V_0^{i+1} \setminus \operatorname{Int} V_0^i$, bounded by $\Sigma_0^{i+1} \cup \Sigma^i_0$, allows us to apply the standard Moser's argument (see [**?**]) and find a family of diffeomorphism $g_s : V \to V$ which are fixed on $\mathcal{O}p \left(\bigcup\limits_1^\infty \Sigma^i_0 \right)$ and such that the composition $h_s : g_s \circ \Psi_s$ is the required symplectomorphism $(V, \omega_0) \to (V, \omega_s)$. □

In particular, Proposition 5.2 implies that *a family* (V, ω_s) *of Liouville manifolds with cylindrical ends consists of symplectomorphic manifolds provided, that the closure* $\overline{\bigcup\limits_s \mathrm{Core}(V, \omega_s)}$ *of the union of their cores is compact.*

5.3 Weinstein manifolds

A symplectic manifold (V, ω) is called *Weinstein* if it admits a complete Liouville field X which is gradient like for an exhausting Morse function $\varphi : V \to \mathbb{R}$. This means that zeroes of X are non-degenerate and coincide with the critical points of φ, and $d\varphi(X) > 0$ outside of the critical locus of φ. The quadruple (V, ω, X, φ) is called a *Weinstein structure* on V. In order to consider deformations of Weinstein structures we will also also allow φ and X to have death-birth (or cusp) singularities. The function φ is called a *Lyapunov function* for X.

A compact symplectic manifold (W, ω) with boundary ∂W is called a *Weinstein domain* if it admits a Liouville vector field transverse to the boundary ∂W which is gradient like for a Morse function $\phi : W \to \mathbb{R}$ which is constant on the boundary. Thus any Weinstein manifold (V, ω, X, ϕ) can be exhausted by Weinstein domains $W_k = \{\phi \leq d_k\}$, where $d_k \uparrow \infty$ is a sequence of regular values of the function ϕ.

A Weinstein manifold (V, ω) is said to be of *finite type* if it admits X with finitely many critical points. Note that by attaching a cylindrical end any Weinstein domain (W, ω, X, ϕ) can be completed to a finite type Weinstein manifold, called its *completion* and denoted by $\mathrm{Compl}(W, \omega, X, \phi)$. Conversely, finite type Weinstein structures can be obtained by attaching a cylindrical end to a Weinstein domain. The contact manifolds which appear as ideal boundaries of finite type Weinstein manifolds, or which is the same, boundaries of Weinstein domains, are called *Weinstein*, or in view of Therem **??** *Stein fillable.*

An important example of a Weinstein structure is provided by the cotangent bundle $V = T^*(M)$ of a closed manifold M, $\omega = d(pdq)$ with the standard symplectic structure $\omega = d\lambda$, $\lambda = pdq$. on V. Take any Riemannian metric on M and a Morse function $f : M \to \mathbb{R}$. Note that the Hamiltonian vector field Y generated by the function $F = p\nabla f$ (or in a more invariant notation $\lambda(\nabla f)$) coincides with ∇f along the zero-section of T^*M. Then the vector $X = p\frac{\partial}{\partial p} + Y$ is Liouville, and has a Morse function $\varphi = \frac{1}{2}|p|^2 + +f(q)$ as its Lyapunov function if f is small enough.

Exercise 5.3. Find explicitly a Weinstein structure on T^*M if M is not compact.

The product of two Weinstein manifolds $(V_1, \omega_1, X_1, \phi_1)$ and $(V_2, \omega_2, X_2, \phi_2)$ has a canonical Weinstein structure $(V_1 \times V_2, \omega_1 \oplus \omega_2, X_1 \oplus X_2, \phi_1 \oplus \phi_2)$. In particular, the product

$$(V, \omega, X, \phi) \times \left(\mathbb{R}^{2k}, \sum dx_i \wedge dx_i, \frac{1}{2} \sum \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}\right), \sum \left(x_i^2 + y_i^2\right)\right)$$

is called the *k-stabilization* of the Weinstein manifold (V, ω, X, ϕ).

Let us recall some standard symplectic geometric notions. Let (V, ω) be a symplectic vector space (V, ω), i.e. is a real vector space V of dimension $2n$ with a nondegenerate skew-symmetric bilinear form ω. To a subspace $W \subset V$ we associate its ω-orthogonal complement

$$W^{\perp_\omega} := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

A subspace $W \subset V$ is called

- *isotropic* if $W \subset W^{\perp_\omega}$,
- *coisotropic* if $W^{\perp_\omega} \subset W$,
- *Lagrangian* if $W^{\perp_\omega} = W$,
- *symplectic* if $W \cap W^{\perp_\omega} = \{0\}$.

Note that $\dim W \leq n$ if W is isotropic, $\dim W \geq n$ if W is coisotropic, and $\dim W = n$ if W is Lagrangian.

The above notions extend in the obvious manner to submanifolds of symplectic manifolds.

To state the next result, call a zero p of a vector field X *hyperbolic* if the linearization of X at p is nondegenerate without purely imaginary eigenvalues.

Proposition 5.4. *Let (V, ω) be a symplectic manifold with an expanding vector field X, and let p be a hyperbolic zero of X. Then*

(a) the stable manifold $W^-(p)$ is isotropic, and

(b) the unstable manifold $W^+(p)$ is coisotropic.

Proof. Let $\phi_t : V \to V$ be the flow of X. Abbreviate $W^+ := W^+(p)$ and $W^- := W^-(p)$, so $T_pV = T_pW^+ \oplus T_pW^-$. All eigenvalues of the linearization of X at p have negative real part on T_pW^- and positive real part on T_pW^+. It follows that the differential $T_p\phi_t : T_xV \to T_{\phi_t(x)}V$ satisfies

$$\lim_{t\to\infty} T_x\phi_t(v) = 0 \text{ for } x \in W^-, v \in T_xW^-,$$
$$\lim_{t\to-\infty} T_x\phi_t(v) = 0 \text{ for } x \in W^+, v \in T_pW^+.$$

(a) Let $x \in W^-$ and $v, w \in T_xW^-$. Since $\phi_t(x) \to p$ as $t \to \infty$, the preceding discussion shows

$$e^t\omega(v,w) = (\phi_t^*\omega)(v,w) = \omega_{\phi_t(x)}(T_x\phi_t \cdot v, T_x\phi_t \cdot w) \mapsto 0$$

as $t \to \infty$. This implies $\omega(v,w) = 0$.

(b) Let $x \in W^+$ and $v \in (T_xW^+)^{\perp_\omega} \subset T_xV$. Suppose $v \notin T_xW^+$. Take a sequence $t_k \to -\infty$ and let $x_k := \phi_{t_k}(x)$. Pick $\lambda_k > 0$ such that $v_k := \lambda_k T_x\phi_{t_k} \cdot v$ has norm 1 with respect to some metric on V. Note that $v_k \in (T_{x_k}W^+)^{\perp_\omega}$ for all k. Pass to a subsequence so that $v_k \to v_\infty \in T_pV$. Since $T_x\phi_{t_k}$ contracts the component of v tangent to W^+ and expands the transverse component, we find $0 \neq v_\infty \in T_pW^-$.

We claim that $v_\infty \in (T_pW^+)^{\perp_\omega}$. Otherwise, there would exist a $w_\infty \in T_pW^+$ with $\omega(v_\infty, w_\infty) \neq 0$. But then $\omega(v_k, w_k) \neq 0$ for k large and some $w_k \in T_{x_k}W^+$, contradicting $v_k \in (T_{x_k}W^+)^{\perp_\omega}$. Hence v_∞ is ω-orthogonal to T_pW^+. Since T_pW^- is isotropic by part (a), v_∞ is also ω-orthogonal to T_pW^-. But this contradicts the nondegeneracy of ω because $T_pV = T_pW^+ \oplus T_pW^-$. □

Corollary 5.5. *Let (V,ω) be a symplectic manifold of dimension $2n$ with an expanding vector field X, and let p be a hyperbolic zero of X. Then the stable manifold satisfies $\dim W^-(p) \leq n$.*

Note that given a Weinstein structure (V, ω, X, ϕ) then any regular level $\Sigma_c = \{\phi = c\}$ carries a canonical contact structure ξ_c. defined by a contact form $\lambda = i(X)\omega|_{\Sigma_c}$.

Lemma 5.6. *Let (V, ω, X, ϕ) be a Weinstein structure.*

(i) If c is a regular value of ϕ then for any critical point $p \in V$ with $\phi(p) > c$ the intersection $W^-(p) \cap \Sigma_c$ is isotropic in a contact sense, i.e. it is tangent to ξ_c.

(ii) Suppose ϕ has no critical values in $[a,b]$. Let $\Lambda^a \subset \Sigma_a = \phi^{-1}(a)$ be an isotropic submanifold. Then the image of Λ^a under the flow of X intersects Σ_b in a totally real submanifold Λ^b.

Proof. (i) $\lambda|_{W^-(p)} = i(X)(\omega|_{W^-(p)} = 0$.

(ii) Let $f > 0$ be the function such that $L_{fX}\phi \equiv 1$ on $\phi^{-1}([a,b])$. Denote by ψ_t the flow of fX, thus $\Lambda^b = \psi_{b-a}(\Lambda^a)$. Since $L_X\omega = \omega$, the 1-form $\lambda := i_X\omega$ satisfies

$$L_X\lambda = (di_X + i_X d)i_X\omega = i_X\omega = \lambda.$$

Hence $L_{fX}\lambda = f\lambda$, so the flow ψ_t only rescales λ and the lemma follows. □

Lemma 5.6 shows that every Weinstein structure on V provides a handlebody decomposition of V where cells attached along contactly isotropic (Legendrian in the maximum index case) spheres. The core discs of the handles are isotropic in the symplectic sense. We will discuss this handlebody decomposition picture with more details in Section **??** below.

5.4 Weinstein structure of a Stein manifold

Proposition 5.7. *[(see EliGro91] Let (V, J) be a Stein manifold and $\phi : V \to \mathbb{R}$ a completely exhausting (see Section 1.3 above) J-convex Morse function. Then*

$$(\omega_\phi = -d^{\mathbb{C}}\phi, X_\phi = \nabla_\phi\phi, \phi)$$

Is a Weinstein structure on V. Symplectic manifold (V, ω_ϕ) is independent up to isotopic to the identity symplectomorphism of the choice of completely exhausting J-convex Morse function ϕ.

Proof. By definition of J-convexity, $\omega_\phi := -dd^{\mathbb{C}}\phi$ is a symplectic form, i.e., a closed nondegenerate 2-form. Denote $X_\phi := \nabla\phi$ the gradient of ϕ taken with respect to the metric $\langle X, Y\rangle := \omega_\phi(X, JY)$. Then X_ϕ is Liouville. Indeed, for any $Y \in TV$ we have

$$d^{\mathbb{C}}\phi(Y) = \langle \nabla\phi, JY\rangle = -\omega_\phi(\nabla\phi, Y) = -i_{X_\phi}\omega_\phi(Y).$$

Hence

$$i_{X_\phi}\omega_\phi = -d^{\mathbb{C}}\phi, \qquad L_{X_\phi}\omega_\phi = \omega_\phi.$$

To prove the second part of the proposition consider two completely exhausting J-convex functions $\phi_0, \phi_1 : V \to \mathbb{R}_+$. Using Lemma 2.18 we find smooth functions $h_0, h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ with $h_0, h_1' \to \infty$ and $h_0'', h_1'' > 0$, a completely exhausting function $\psi : V \to \mathbb{R}_+$, and a sequence of compact domains V_k, $k = 1, \dots,$ with smooth boundaries $\Sigma_k = \partial V_k$, such that

- $V_k \subset \operatorname{Int} V_{k+1}$ for all $k \geq 1$;
- $\bigcup\limits_k V_k = V$;
- Σ_{2j-1} are level sets of the function ϕ_1 and Σ_{2j} are level sets of the function ϕ_0,$j = 1, \dots;$

- $\psi = h_0 \circ \phi_0$ on $\mathcal{O}p\left(\bigcup_{j=1}^{\infty} \Sigma_{2j-1}\right)$ and $\psi = h_1 \circ \phi_1$ on $\mathcal{O}p\left(\bigcup_{j=1}^{\infty} \Sigma_{2j}\right)$.

Let us construct now a compact type homotopy between the Weinstein structures $(\omega_{\phi_0}, X_{\phi_0}, \phi_0)$ and $(\omega_{\phi_1}, X_{\phi_1}, \phi_1)$ on V. Then Proposition 5.2 will imply that the symplectic manifolds (V, ω_{ϕ_0}) and (V, ω_{ϕ_1}) are symplectomorphic via an isotopic to the identity diffeomorphism.

The required compact type homotopy can now be be constructed as a composition of four *elementary* compact type homotopies. First, note that for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h' \to \infty$ and $h'' > 0$ the linear combination $h^s(x) = (1-s)x + sh(x)$ has the same properties for any $s \in (0, 1]$, and hence the Weinstein structures which correspond to the family of completely exhausting J-convex functions $h_i^s \circ \phi_i$ provide elementary compact type homotopies between the Weinstein structures $(\omega_{\phi_i}, X_{\phi_i}, \phi_i$ and $(\omega_{h_i \circ \phi_i}, X_{h_i \circ \phi_i}, h_i \circ \phi_i$, $i = 0, 1$. On the other hand, for each $i = 0, 1$ the family $\phi_i^s = (1-t)h_i \circ \phi_i + t\psi$, $s \in [0, 1]$, consists of exhausting J-convex functions which concide near boundaries of an exhausting sequence of compact domains. Without loss of generality we can also assume that these functions are completely exhausting. Hence the Weinstein structures which they generate provide elementary homotopies between $(\omega_{h_i \circ \phi_i}, X_{h_i \circ \phi_i}, h_i \circ \phi_i$ and $(\omega_\psi, X_\psi, \psi)$. □

Note that the contact structure ξ_c defined on a regular level set Σ_c by the form $d\phi^{\mathbb{C}}|_{\Sigma_c}$ is formed in this case by the distribution of complex tangent hyperplanes to the J-convex hypersurface Σ_c.

Remark 5.8. Let (V, J) be any almost complex manifold which admits an exhausting J-convex Morse function $\phi : V \to \mathbb{R}$. Then even if the symplectic form $\omega_\phi = -dd^{\mathbb{C}}\phi$ is not compatible with J, one still gets a Weinstein structure $(V, \omega_\phi, X_\phi, \phi)$, similar to the one defined in Proposition 5.7. The only difference in this case that the necessary Liouville vector field X_ϕ should be defined directly as ω_ϕ-dual to $-D^{\mathbb{C}}\phi$ and not as the gradient of ϕ. Nevertheless, X_ϕ in this case is gradient-like for ϕ. Indeed, let $p \in V$ be a regular point of ϕ. We need to check that $X = X_\phi(p) \notin T_p$, where T_p is the tangent to the level set of ϕ ant p. Note that we have $JX \in T_p$. Indeed,

$$d\phi(JX) = d^{\mathbb{C}}\phi(X) = dd^{\mathbb{C}}\phi(X, X) = 0$$

Hence, if we have also $X \in T_p$, then $X, JX \in \xi_p = T_p \cap JT_p$. On the other hand, for any vector $Y \in \xi_p$ we have $\omega_\phi(X, Y) = -d^{\mathbb{C}}\phi(Y) = 0$, i.e. $X \in \mathrm{Ker}\omega_\phi|_{\xi_p}$, which contradicts to J-convexity of ϕ. Completeness of X can be achieved similarly to the integrable case.

Proposition 5.7, Remark 5.8 and Corollary 5.8 imply

Corollary 5.9. *Indices of critical points of any J-convex function on a $2n$-dimensional almost complex manifold are $\leq n$.*

Chapter 6

Weinstein handlebodies

6.1 Handles in the smooth category

For integers $0 \leq k \leq m$ and a number $\varepsilon > 0$ consider the *m-dimensional k-handle*

$$H := H^k := H^k_\varepsilon := D^k_{1+\varepsilon} \times D^{m-k}_\varepsilon,$$

where D^k_r denotes the closed k-disk of radius r. We will use the following notations:

- the *core disk* $D := D^k_1 \times \{0\}$ and the *core sphere* $S := \partial D$;
- the *lower boundary* $\partial^- H := \partial D^k_1 \times D^{m-k}_\varepsilon$;
- the *upper boundary* $\partial^+ H := D^k_1 \times \partial D^{m-k}_\varepsilon$;
- the normal bundle $\nu := T(\partial^- H)|_S = S \times \mathbb{R}^{m-k}$ to S in $\partial^- H$;
- the outward normal vector field η along $S \subset D$;
- the *attaching region* $U := H \setminus D^k_1 \times D^{m-k}_\varepsilon$.

We are not fixing the "width" ϵ of the handle and allow us to choose it as small as it is convenient.

Now let W be a compact m-manifold with boundary ∂W. An *attaching map* for a k-handle is an embedding $f : \partial^- H \hookrightarrow \partial W$. Extend f to an embedding $F : (U, U \cap \partial^- H) \hookrightarrow (W, \partial W)$ by mapping η to an inward pointing vector field along ∂W. Then we can *attach a k-handle* to W by the map f to get a manifold

$$W \cup_f H := W \amalg H/_{H \leq x \sim F(x) \in W}.$$

Different extensions F give rise to manifolds that are *canonically diffeomorphic*, i.e., related by a diffeomorphism that is unique up to isotopy. Moreover, the diffeomorphism equals the identity on a *shrinking of* W, i.e., the complement of a tubular neighbourhood of ∂W.

Remark 6.1. Note that the boundary of $W \cup_f H$ has a corner along $f(\partial D_1^k \times \partial D_\varepsilon^{m-k})$. But this corner can be smoothed in a canonical way as follows (cf. Chapter 7): Introduce the norms

$$R := \sqrt{x_1^2 + \cdots + x_k^2} \quad \text{and} \quad r := \sqrt{x_{k+1}^2 + \cdots + x_m^2}.$$

Pick a concave curve γ in the first quadrant of the (r, R)-plane as in Figure ??? [to be added] which equals the curve $R \equiv 1$ near $(\varepsilon, 1)$ and $r \equiv \delta$ near $(\delta, 0)$ for some $0 < \delta < \varepsilon$. Denote by $H_\gamma \subset H$ the region bounded by the hypersurface $\{(r, R) \in \gamma\}$ and containing the core disk. Then $W \cup_f H_\gamma$ is a smooth manifold with boundary which is easily seen to be independent of the curve γ, up to canonical diffeomorphism fixed on a shrinking of W. Therefore, we will suppress γ from the notation and denote the resulting smooth manifold with boundary again by $W \cup_f H$.

In particular, this argument shows independence of the "width" ε.

Remark 6.2. The boundary of $W \cup_f H$ is obtained from ∂W by *surgery of index* k, i.e., by cutting out a copy of $\partial D^k \times D^{m-k}$ and gluing in $D^k \times \partial D^{m-k}$ along the common boundary $\partial D^k \times \partial D^{m-k}$. The manifold $W \cup_f H \setminus W'$, where $W' \subset W$ is the complement of a tubular neighbourhood of ∂W, provides a canonical cobordism between ∂W and $\partial(W \cup_f H)$. This cobordism carries a Morse function constant on the boundaries and with a unique critical point of index k in the center of the handle, see [37].

Remark 6.3. By the tubular neighbourhood theorem (see [33]), the attaching map $f : \partial^- H \hookrightarrow W$ is uniquely determined, up to isotopy, by the following two data:

- the embedding $f|_S : S \cong S^{k-1} \hookrightarrow \partial W$ (the *attaching sphere*);
- the trivialization $df : \nu \cong S \times \mathbb{R}^{m-k} \to \nu_f$ of the normal bundle to f in ∂W (the *normal framing*).

Lemma 6.4. *An isotopy of attaching maps* $f_t : \partial^- H \hookrightarrow \partial W$, $t \in [0, 1]$, *induces a canonical family of diffeomorphisms* $\phi_t : W \cup_{f_0} H \to W \cup_{f_t} H$.

Proof. By the isotopy extension theorem (see [33]), (after possibly shrinking ε) there exists an isotopy of diffeomorphisms $\psi_t : \partial W \to \partial W$ such that $\psi_0 = 1\!\!1$ and $f_t = \psi_t \circ f_0$. Let $\partial W \times [-1, 0]$ be a collar neighbourhood of $\partial W \cong \partial W \times \{0\}$ and define for each t a diffeomorphism

$$\Psi_t : \partial W \times [-t, 0] \to \partial W \times [-t, 0], \qquad (x, \tau) \mapsto \big(\psi_{\tau+t}(x), \tau\big).$$

Then Ψ_t fits together with the identity on $W \setminus (\partial W \times [-t, 0])$ and H to a diffeomorphism $\phi_t : W \cup_{f_0} H \to W \cup_{f_t} H$, see Figure ??? [to be added]. □

Example 6.5. In general, the diffeomorphism type of $W_{(f,\beta)}H$ depends on the normal framing. It also generally depends on the particular parametrization $f : S \to f(S)$ of the embedded sphere $f(S) \subset \partial W$. For example, attaching an m-handle to the m-ball D^m via a diffeomorphism $f : S^{m-1} \to S^{m-1}$ yields a manifold $D^m \cup_f H$ that is easily seen to be homeomorphic to S^m. However, it is in general not diffeomorphic to S^m. Indeed, by Lemma 6.4, $f \mapsto D^m \cup_f H$ defines a map from isotopy classes of diffeomorphisms of S^{m-1} to smooth structures on S^m (up to diffeomorphism). This map is known to be surjective for all $m \neq 4$ (see [33]; the remaining case $m = 4$ amounts to the 4-dimensional smooth Poincaré conjecture). E.g., all the 28 smooth structures on S^7 arise in this way.

Morse theory. For a function $\phi : V \to \mathbb{R}$ on a manifold and $c < d$ we introduce the following obvious notations:

$$V^c := \phi^{-1}(c), \qquad V^{\leq c} := \phi^{-1}\big((-\infty, c]\big), \qquad V^{[c,d]} := \phi^{-1}([c,d]) \qquad \text{etc.}$$

The main result of Morse theory can now be formulated as follows (see [37]):

Proposition 6.6. *Let $\phi : V \to \mathbb{R}$ be a proper function on a manifold such that $V^{[a,b]}$ contains a unique nondegenerate critical point p on level $c \in (a,b)$. Then $V^{\leq b}$ is obtained from $V^{\leq a}$ by attaching a k-handle, where $k = \mathrm{ind}(p)$.*

Since every (paracompact) manifold admits an exhausting Morse function with distinct critical levels (i.e., every level contains at most one critical point) (see [37]), this implies

Corollary 6.7. *Every manifold is obtained from a ball by successive attaching of at most countably many handles.*

We will later need the following lemma about equivalence of Morse functions.

Lemma 6.8. *Let $W^n \subset V^n$ be compact manifolds with boundary and $\Delta \subset V \setminus W$ be an embedded k-disk transversely attached to W along its boundary. Let $\phi, \psi : V \to \mathbb{R}$ be two Morse functions with a unique index k critical point $p \in \Delta$ and regular level sets $\partial W = \phi^{-1}(a) = \psi^{-1}(a)$ and $\partial V = \phi^{-1}(b) = \psi^{-1}(b)$, $a < b$. Suppose that $\phi = \psi$ on $W \cup \Delta$ and their restrictions to Δ have a nondegenerate maximum at p. Then there exists a diffeomorphism $f : V \to V$ with $f|_{W\cup\Delta} = \mathbb{1}$, isotopic to $\mathbb{1}$ rel $W \cup \Delta$, such that $f^*\psi = \phi$.*

Proof. By the Morse lemma, there exists an orientation preserving diffeomorphism $g : U \to U'$ between neighbourhoods of p such that $g^*\psi = \phi$. Moreover, we may assume the $g = \mathbb{1}$ on $U \cap \Delta$. (To see this, first find coordinates $x_1, \dots, x_k$ on Δ near p in which $\phi(x) = -x_1^2 - \cdots - x_k^2$ and extend them to coordinates $x_1, \dots, x_n$ for V near p. Then apply the proof of the Morse lemma in [36] to find new coordinates $u_1, \dots, u_n$ near p in which $\phi(u) = -u_1^2 - \cdots - u_k^2 + u_{k+1}^2 \cdots + u_n^2$. Inspection of the proof shows that $u_i = x_i$ on Δ. Pick corresponding coordinates v_i for ψ and define g by $u_i \to v_i$.) After shrinking U, U', we can extend g to a diffeomorphism $g : B \to B$ of a ball B containing U, U' such that $g = \mathbb{1}$ near ∂B

and g is isotopic to $\mathbb{1}$ rel ∂B. Extend g to a diffeomorphism $g : W \cup N \to W \cup N'$, where N, N' are neighbourhoods of Δ in V, such that $g = \mathbb{1}$ outside B. Using the flow of a gradient-like vector field for ψ on $N' \setminus U'$, we can modify g by an isotopy fixed on $W \cup \Delta \cup U$ to a diffeomorphism h satisfying $h^*\psi = \phi$. Now pick a gradient-like vector field X for ϕ on $N \setminus U$, tangent to ∂N, and set $X' := h_* X$ on $N' \setminus U'$. Extend X to a gradient-like vector field on $V \setminus (W \cup U)$ and normalize it such that $X \cdot \phi = 1$, similarly for X'. Denote the flows of X, X' by γ_t, γ'_t. For $x \in V \setminus (W \cup U)$, let $t(x) < 0$ be the unique time for which $\gamma_{t(x)}(x) \in \partial W$. Now define $f : V \to V$ by $f := h$ on $W \cup U$ and $f(x) := \gamma'_{-t(x)} \circ \gamma_{t(x)}(x)$ on $V \setminus (W \cup U)$. □

6.2 Weinstein handlebodies

6.2.1 Standard handle

We will be interested only in attaching handles of index $k \leq n$ and view the handle $H = H(\varepsilon) = D^k_{1+\varepsilon} \times D^{2n-k}_{\varepsilon}$ as canonically embedded in $\mathbb{C}^n$ as the bidisk

$$\{\sum_{j=1}^{k} x_j^2 \leq (1+\varepsilon)^2, \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{n} |z_j|^2 \leq \epsilon^2\}, \tag{6.1}$$

where $z_i = x_j + iy_j$, $j = 1, \ldots, n$, are the complex coordinates in $\mathbb{C}^n$. In particular, the handle H carries the *standard complex structure* i, as well as the *standard symplectic structure* $\omega_{\rm st}$.

The symplectic form $\omega_{\rm st}$ on H admits a hyperbolic Liouville field

$$X_{\rm st} = \sum_1^k \left(x_j \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial y_j} \right) + \frac{1}{2} \sum_{k+1}^{n} \left(x_l \frac{\partial}{\partial x_l} + y_l \frac{\partial}{\partial y_l} \right). \tag{6.2}$$

Let us denote by ξ^- the contact structure $\alpha_{\rm st}|_{\partial^- H} = 0$ defined on $\partial^- H$ by the Liouville form $\alpha_{\rm st} = i(X_{\rm st})\omega_{\rm st}$. Note that the bundle $\xi_-|_S$ canonically splits as $TS \otimes \mathbb{C} \oplus \varepsilon^{n-k}$, where ε^{n-k} is a trivial $(n-k)$-dimensional complex bundle. We will denote by σ_S the isomorphism

$$TS \otimes \mathbb{C} \oplus \varepsilon^{n-k} \to \xi_-|S.$$

Suppose we are given a real k-dimensional bundle λ, a complex n-dimensional bundle E, $n \geq k$, and an injective totally real homomorphism $\phi : \lambda \to E$. Then ϕ canonically extends to a complex homomorphism $\phi \otimes \mathbb{C} : \lambda \otimes \mathbb{C} \to E$. If $n > k$ and $\phi \otimes \mathbb{C}$ extends to a fiberwise complex isomorphism $\Phi : \lambda \otimes \mathbb{C} \oplus \varepsilon^{n-k}$ then Φ is called a *saturation* of λ. When $n = k$ the saturation is unique.

Let (V, ω, X, ϕ) be a Weinstein manifold, p a critical point of index k of the function ϕ, $a < b = \phi(p)$ a regular value of ϕ. Denote $W := \{\phi \leq a\}$. Suppose that the stable manifold of p intersects $V \setminus \operatorname{Int} W$ along a disc D, and $S = \partial D$ be the attaching sphere. The inclusion $TS \hookrightarrow \xi$ extends canonically to an injective complex homomorphism $TS \otimes \mathbb{C} \to \xi$, while the inclusion $TD \hookrightarrow TV$ extends to an injective complex homomorphism $TD \otimes \mathbb{C} \to TV$. There exists a homotopically unique complex trivialization of the normal bundle to $TS \otimes \mathbb{C}$ in ξ which extends to D as a trivialization of the normal bundle to $TD \otimes \mathbb{C}$ in TV. This trivialization provides a canonical isomorphism $\Phi_D : TS \otimes \mathbb{C} \oplus \varepsilon^{n-k} \to \xi|_S$. We will call S *canonical saturation* of the inclusion $S \hookrightarrow \partial W$.

6.2.2 Attaching a handle to a Weinstein domain

Proposition 6.9. *Let (W, ω, X, ϕ) be a $2n$-dimensional Weinstein domain with boundary ∂W and ξ the induced contact structure $\{\alpha|_{\partial W} = 0\}$ on ∂W defined by the Liouville form $\alpha = i(X)\omega$. Let $h : S \to \partial W$ be an isotropic embedding of the $(k-1)$-sphere S covered by a saturation $\Phi : TS \otimes \mathbb{C} \oplus \varepsilon^{n-k} \to \xi$ of the differential $dh : TS \to \xi$. Then there exists a Weinstein domain $(\widetilde{W}, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ such that $W \subset \operatorname{Int} \widetilde{W}$, and*

- *(i)* $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})|_W = (\omega, X, \phi)$;
- *(ii) the function $\widetilde{\phi}|_{\widetilde{W} \setminus \operatorname{Int} W}$ has a unique critical point p of index k.*
- *(iii) the stable disc D of the critical point p is attached to ∂W along the sphere $h(S)$, and the canonical saturation $\Phi|_D$ coincides with Φ.*

Given any two Weinstein extensions $(W_0, \omega_0, X_0, \phi_0)$ and $(W_1, \omega_1, \underline{X}_1, \phi_1)$ of (W, ω, X, ϕ) which satisfy properties (i)-(iii), there exits a fixed on W diffeomorphism $g : W_0 \to W_1$ such that (ω_0, X_0, ϕ_0) and the pull-back structure $(g^*\omega_1, g^*X_1, \phi_1 \circ g)$ are homotopic in the class of Weinstein structures which satisfy conditions (i)-(iii). In particular, the completions $\operatorname{Compl}(W_0, \omega_0, X_0, \phi_0)$ *and* $\operatorname{Compl}(W_1, \omega_1, X_1, \phi_1)$ *are symplectomorphic via a symplectomorphism fixed on W.*

We say that the Weinstein domain $(\widetilde{W}, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ is obtained from (W, ω, X, ϕ) by attaching a handle of index k along an isotropic sphere $h : S \to \partial W$ with a saturation homomorphism Φ.

Proof. We will assume that $\phi|_{\partial W} = 1$. Let $H = H(\varepsilon)$ be the standard handle given by (6.1). We will extend the structure of Weinstein domain to $W \underset{h}{\cup} H$ as follows. First extend (ω, X, ϕ) to an open domain W' which contains W. In particular, $\{\phi \leq 1 + \delta\} \subset W'$ is compact for some small $\delta > 0$. By Lemma **??** the isotropic embedding h extends to a diffeomorphism $\widetilde{h}$ of a neighborhood $\mathcal{O}p\, S$ of S in $H(\varepsilon)$ to a neighborhood $\mathcal{O}p\,(h(S))$ of $h(S)$ in W' such that:

- $\widetilde{h}(\partial^- H \cap \mathcal{O}p\, S) = \partial W \cap \mathcal{O}p\,(h(S))$;

- $\widetilde{h}$ induces an isomorphism between the standard Liouville structure $(\omega_{\rm st}, X_{\rm st})$ on $\mathcal{O}p\,(S) \subset H$ and the Liouville structure (ω, X) on $\mathcal{O}p\,(h(S)) \subset W'$;
- the homomorphism $d\widetilde{h} \circ \sigma_S : TS \otimes S \oplus \varepsilon^{n-k} \to \xi$ coincides with the saturation Φ.

Hence, for a sufficiently small $\varepsilon > 0$ the Liouville structures on W' and $H(\varepsilon)$ can be glued together to a Liouville structure $(\widetilde{\omega}, \widetilde{X})$ on $W' \underset{\widetilde{h}}{\cup} H(\varepsilon)$. Next, we will extend the function ϕ to the attached handle. To continue we need the following

Lemma 6.10. *Let X be any Liouville vector field for the standard symplectic structure $\omega_{\rm st}$ on $H(\varepsilon)$ which has a simple hyperbolic zero and the origin and for which the core disc D serves as its stable manifold. Let ψ be a smooth function on a neighborhood $\mathcal{O}p\,S$ of the core sphere $S = \partial D$, $D = D^k_1 \times 0$, such that $\psi|_S = 1$ and $d\psi(X) > 0$. Then for any sufficiently small $\varepsilon, \delta > 0$ there exists a Morse function $\widetilde{\psi} : H(\varepsilon) \to \mathbb{R}$ such that*

(i) $\widetilde{\psi}$ *has a unique critical point (of index k) at the center* 0 *of the handle and* $\widetilde{d}(X) > 0$ *elsewhere;*

(ii) $\widetilde{\psi}$ *coincides with* ψ *on* $\mathcal{O}p\,\big((\{1-\delta \le \psi \le 1+\delta\} \cap \partial_+ H_+(\varepsilon)) \cup \{\psi \le 0\}\big)$;

(iii) $\widetilde{\psi}|_D < 1 + \delta$.

The statement also holds in a parametric form.

Proof of Lemma 6.10. To be added. □

Let us apply Lemma 6.10 to the function $\psi = \phi \circ \widetilde{h}$ on $\mathcal{O}p\,S$. and denote by $\widetilde{\phi}$ the function on $W' \underset{\widetilde{h}}{\cup} H(\varepsilon)$ glued from ϕ on W' and $\widetilde{\psi}$ on $H(\varepsilon)$. Set $\widetilde{\Omega} := \{\widetilde{\phi} \le 1 + \delta\} \subset W' \underset{\widetilde{h}}{\cup} H(\varepsilon)$. Then $(\widetilde{W}, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ is a Weinstein domain with the required properties.

To prove the uniqueness statement let us consider two Weinstein domain structures (ω_0, X_0, ϕ_0) and (ω_1, X_1, ϕ_1) on $\widetilde{W}$ which satisfies conditions (i)-(iii) from 6.9. Let p_0 and p_1 be the critical points for Φ_0 and Φ_1, and D_0 and D_1 be their stable manifolds. The identity map $W \to W$ extends as a diffeomorphism $g : \mathcal{O}p\,(W \cup D_0) \to \mathcal{O}p\,(W \cup D_1)$, which sends D_0 to D_1. According to Lemma **??** we can assume that g is a symplectomorphism. Let us denote by $\widetilde{X}_1$ the pull-back Liouville field $g^* X_1$. Then there exists a homotopy of Liouville fields Y_t, $t \in \mathbb{R}$, connecting $Y_0 = X_0$ and $Y_1 = \widetilde{X}_1$ which satisfies conditions of Lemma 6.10. Hence, Lemma 6.10 provides us with a family ψ_t of extensions of the function ϕ from $W \subset W_0$ to $\mathcal{O}p\,(W \cup D_0) \subset W_0$ as Lyapunov functions for Y_t,

$t \in [0,1]$. Therefore, for a sufficiently small $\delta > 0$ we get a family of Weinstein domains

$$(\widetilde{W}_t := \{\psi_t \le 1+\delta\}, \omega_0, Y_t, \psi_t).$$

Finally note that all the trajectories of the vector field X_0 on $W_0 \setminus \operatorname{Int} \widetilde{W}_0$ transversely enter the domain through $\partial \widetilde{W}_0$ and exit through ∂W_0, and hence the Liouville structures $(\widetilde{W}_0, \omega_0, X_0, \psi_0)$ and $(W_0, \omega_0, X_0, \phi_0)$ are homotopic. Hence, ψ_0 can be extended to W_0 without critical points to $W_0 \setminus \operatorname{Int} W_0$, so it coincides with ϕ_0 near ∂W_0. Therefore, the Weinstein structures $(W_0, \omega_0, X_0, \phi_0)$ and $(\widetilde{W}_0, \omega_0, X_0, \psi_0)$ are homotopic. Similarly, there exists a homotopy between $(W_1, \omega_1, X_1, \phi_1)$ and $(g(\widetilde{W}_1), \omega_1, X_1, \psi_1 \circ g^{-1})$. This completes the proof. □

Remark 6.11. Note that Proposition 6.9 implies that even in the case of infinitely many handles the handlebody description determines the symplectomorphism type of Weinstein manifold. Indeed, it follows that given 2 manifolds $(V_1, \omega_1, X_1, \phi_1)$ and $(V_1, \omega_2, X_2, \phi_2))$ with the same handlebody description, there is a symplectomorphism of a neighborhood U_1 of the isotropic skeleton K_1 of the first manifold onto a neighborhood U_2 of the skeleton K_2 of the second. Moreover, the neighborhoods can b chosen i such a way that their boundaries are transversal to the Liouville fields X_1 and X_2 respectively. On the other hand, $\bigcup\limits_t X_1^t(U_1) = V_1$ and $\bigcup\limits_t X_2^t(U_2) = V_2$, and hence the symplectomorphism $U_1 \to U_2$ can be extended to a symplectomorphism $V_1 \to V_2$ by matching the corresponding trajectories of the Liouville fields.

Theorem 6.12. *Suppose that a $2n$-dimensional almost complex manifold (V, J) admits an exhausting Morse function with critical points of index $\le n$. Then there exists a Weinstein structure (ω, X, ϕ) on V such that the homotopy class $[J_\omega]$ of an almost complex structure compatible with ω coincides with $[J]$.*

Proof. Let $\phi : V \to \mathbb{R}^+$ be an exhausting Morse function with critical points of index $\le n$. The critical values of ϕ are discrete. Let us order them: $c_0 = 0 < c_1 < c_2 \dots$, introduce intermediate regular values $d_k = c_{k-1} + \frac{c_k - c_{k-1}}{2}$, $k = 1, \dots$, and set $W_k := \{\phi \le d_k\}, \Sigma_k := \partial W_k, k = 1, \dots$. Note that there are only finitely many critical point on each critical level. We are going to construct the Weinstein structure inductively on W_k.

W_1 is a disjoint union of finitely many balls. We choose on each of them a Weinstein structure which consists of the standard symplectic structure of the unit ball in the standard symplectic $(\mathbb{R}^{2n}, \sum\limits_1^n dx_k \wedge dy_k)$, the radial Liouville field

$$X = \frac{1}{2}\left(\sum_1^n x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k}\right),$$

and the function ϕ, which can be assumed to be equal $d_1 \left(\sum\limits_1^k x_k^2 + y_k^2\right)$ on each of the balls. We can deform J on V so it becomes compatible with the chosen symplectic form on W_1.

Let us assume that we already constructed the required Weinstein structure (ω, X, ϕ) on W_l for some $l \geq 1$, so that $J|_{W_l}$ is compatible with the Weinstein structure on W_l. The standard Morse theory tells us that W_{l+1} can be obtained from W_l by a simultaneous attaching of several handles of index $\leq n$. Without a loss of generality we can assume that there is just one handle.

Let p be the corresponding critical point of the function ϕ, and Δ the intersection of its stable manifold (formed by the trajectories of X converging to p) with $V \setminus \operatorname{Int} W_l$. Then Δ is a disc of dimension $k = \operatorname{ind} p$, transversely attached to W_l in $W_{l+1} \subset V$. By Proposition 4.14 there exists an isotopy of Δ in $W_{l+1} \setminus \operatorname{Int} W_l$ into a totally real disc Δ' which is J-orthogonally attached to W_l along an isotropic submanifold of ∂W_l.

By Proposition 6.9 we can extend the Liuoville structure from W_l to a Lioville structure (ω', X', ϕ') on a domain $W'_{l+1} \subset \mathcal{O}p\,(W_l \cup \Delta' \subset W_{l+1}$, so that W'_{l+1} is obtained from W_l by attaching a handle of index k with the core disc Δ' using the canonical saturation of the attaching map provided by the totally real disc Δ'. In particular the almost complex structure on W'_{l+1} can be deformed to become compatible with ω''. Now observe that by construction there is an isotopy $\alpha_t : W'_{l+1} \to V$, $t \in [0,1]$, such that α_0 is the inclusion $W'_{l+1} \hookrightarrow V$, $\alpha_1(W'_{l+1}) = W_{l+1}$ and $\alpha_t|_{\mathcal{O}p\, W_l} = \mathrm{Id}$. Moreover, one can arrange that the function $\phi' \circ h$ differs from ϕ by a reparameterization of the image, i.e. $\phi = \beta \circ \phi'$ for a diffeomorphism $\beta : \mathbb{R}_+ \to \mathbb{R}_+$. The push-forward almost complex structure $(\alpha_1)_* J$ extends in the same homotopy class to V. Hence, $(\alpha_1)_*\omega', (\alpha_1)_* X', \phi)$ is the required extension of the Weinstein structure (ω, X, ϕ, ϕ) from W_l to W_{l+1}.

If the function ϕ has finitely many critical points then to complete the proof it remains to attach a cylindrical end to W_N where c_{N-1} is the last critical level. If there are infinitely many point that the resulted structure on $V = W_\infty = \bigcup\limits_1^\infty W_l$ is automatically Weinstein provided that the Liouville vector field X is complete. However, this can be easily achieved by an appropriate rescaling of ω and X in the neighborhood of all regular levels $\partial W_l = \{\phi = d_l\}$, $l = 1, \dots, \infty$. □

Note that if $k < n$, then the standard $2n$-dimensional handle $H(\mathbf{e})$ of index k contains the standard $(2n-2)$-dimensional handle $H'(\varepsilon) = H(\varepsilon) \cap \mathbb{C}^{n-1}$. The contact structure ξ_n^- on $\partial^- H$ canonically splits as $\xi_{n-1}^- \oplus \varepsilon^1$, where ξ_{n-1}^- is the canonical contact structure on $\partial^- H'$. In the next section we will need the following

Lemma 6.13. *Let (W, ω, X, ϕ) be a Weinstein domain of dimension $2n$, and W' a codimension 2 submanifold which is invariant with respect to X, and such that the restriction $(\omega|_{W'}, X|_{W'}, \phi|_{W'})$ defines on W' a Weinstein domain structure. Suppose that the normal bundle to W' in W is trivial. Let $h : S \to \partial W'$ be an isotropic embedding together with a saturation homomorphism $\Phi' : TS \otimes \mathbb{C} \oplus \mathbb{C}^{n-k-1} \to \xi'$. then one can simultaneously attach the handle H' to W' using h and Φ', and the handle H to W using h and $\Phi = \phi' \oplus \mathrm{Id}$:*

$TS \otimes \mathbb{C} \oplus \mathbb{C}^{n-k-1} \oplus \varepsilon^1 \to \xi|\partial W' = \xi' \oplus \varepsilon^1$ *to get a pair a Weinstein domains* $(\widetilde{W} = W \underset{h,\Phi}{\cup} H, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ *and* $(\widetilde{W}' = W' \underset{h,\Phi}{\cup} H', \widetilde{\omega}', \widetilde{X}, \widetilde{\phi}')$ *such that*

- $(\widetilde{W}', \partial\widetilde{W}') \subset (\widetilde{W}, \widetilde{\partial}W)$, $\widetilde{X}'$ *is tangent to* $\widetilde{W}'$;
- $(\widetilde{\omega}', \widetilde{X}', \widetilde{\phi}') = (\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})|_{\widetilde{W}'}$, *and*
- *the normal bundle to* $\widetilde{W}'$ *in* $\widetilde{W}$ *is trivial.*

6.3 Subcritical Weinstein manifolds

A $2n$-dimensional Weinstein manifold (V, ω) is called *subcritical* if it admits a Weinstein structure (X, ϕ) such that all critical points of the function ϕ have index $< n$. More precisely, it is called *k-subcritical*, $k \geq 1$ if all critical points of ϕ have index $\leq n - k$.

Theorem 6.14 (Cieliebak, [**?**]). *Let (V, ω, X, ϕ) be a k-subcritical $2n$-dimesional Weinstein manifold. Then (V, ω) is symplectomorphic to the k-stabilization of a Weinstein manifold (V', ω', X', ϕ') of dimension $2(n-k)$.*

We will begin with the following

Lemma 6.15. *Suppose that (M, ξ) be a $(2n+1)$-dimensional contact manifold and (N, ζ) its codimension 2 contact submanifold with a trivial normal bundle. Let S be a k-dimensional manifold, $k < n$, $f : S \to V$ an isotropic embedding and $\Phi : E = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k} \to \xi$ a saturation of its differential $df : TS \to \xi$. Suppose that there exists a homotopy $f_t : S \to M, t \in [0,1]$, which begins with $f_1 = f$ and ends at a map $f_1 : S \to N$. Then there exists an isotropic isotopy $g_t : S \to M$ and a family Φ_t of saturation of $df_t, t \in [0,1]$, such that*

- $g_0 = f$;
- $g_1(S) \subset N$;
- g_t *is* C^0*-close to* f_t.
- *the restriction of* Φ_1 *to* $E_1 = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k-1} \subset E = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k}$ *is a saturation of the homomorphism* $dg_1 : S \to \zeta$.

Proof. By assumption, there is a splitting $\mu : \zeta \oplus \varepsilon^1 \to \xi|_N$, where ε^1 is a trivial complex bundle. Denote by ν the vector field $\mu(\mathbf{e})$ where $\mathbf{e}$ generates ε^1. Consider a homotopy $\Psi_t : TS \otimes \mathbb{C} \oplus \varepsilon^{n-k} \to \xi$ of complex homomorphisms which covers the homotopy f_t, $t \in [0,1]$, and begins with $\Psi_0 = \Phi$. We can assume that $\Psi_1(\mathbf{e}_{n-k}) = \nu$, where $\mathbf{e}_{n-k}$ is the generator of the second summand in the decomposition $\varepsilon^{n-k} = \varepsilon^{n-k-1} \oplus \varepsilon^1$. Indeed, the obstructions to do that lie in the groups $\pi_j(S^{2n-1})$, $j \leq k < n$, which are trivial for any $n > 1$.

Hence, we can further adjust Φ_t to ensure that $\Psi_1|_{E_1}$ is a saturation of a totally real homomorphism $\psi : TS \to \zeta$. Now we apply Gromov's h-principle for isotropic immersions 4.8 to C^0-approximate the map $f_1 : S \to N$ by an isotropic immersion $\widetilde{f_1} : S \to N$, whose differential $d\widetilde{f_1} : TS \to \zeta$ is homotopic to ψ through totally real homomorphisms $TS \to \zeta$. Note that the homotopy of complex homomorphisms Ψ_t can be modified into $\widetilde{\Psi}_t : E \to \xi$ so that it ends at a saturation $\widetilde{\Psi}_1 : TS \to \xi$ of the homomorphism $d\widetilde{f_1}$ such that $\widetilde{\Psi}_1(E_1) \subset \zeta$. Next, we apply again Theorem 4.8 and construct an isotropic regular homotopy g_t, $t \in [0,1]$, connecting $g_0 = f$ with $g_1 = \widetilde{f_1}$, together with a family $\Phi_t : E \to \xi$ of saturations of dg_t such that the paths $\widetilde{\Psi}_t$ and Φ_t, $t \in [0,1]$, are homotopic with fixed ends. It remains to note that by dimensional reasons (see Lemma 4.10) we can assume that g_t is an isotopy, rather than a regular homotopy. □

Proof of Theorem 6.14. It is sufficient to consider the case $k = 1$. As in the proof of Theorem 6.12 let $c_0 < c_1 \dots$ be the critical levels of the function ϕ, $d_1 < \dots$ intermediate regular values: $c_1 < d_1 < c_1 < d_2 < \dots$ and $W_l = \{\phi \le d_l\}$, $l = 1, \dots$. We will construct the required Weinstein manifold $V' \subset V$ inductively by successively adjusting the handlebody decomposition of V. On each step we will change the Weinstein domain structure on W_k by Weinstein homotopy, and change the attaching map by contact isotopies of ∂W_k. As it is explained above this will not affect the symplectomorphism type of the resulted Weinstein manifold.

Up to Weinstein homotopy we can assume that W_1 is a round ball in $\mathbb{C}^n = \mathbb{R}^{2n}$ with the standard symplectic structure and the radial Liouville field. We set $W_1' := W_1 \cup \mathbb{C}^{n-1}$. Suppose we already deformed a Weinstein domain structure on W_l, so that for the resulted Liouville structure $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi}$ there exists a codimension 2 submanifold with boundary $(W_l', \partial W_l') \subset (W_l, \partial W_l)$ such that

a) $\widetilde{X}$ is tangent to W_l';

b) the function $\widetilde{\phi}$ has no critical points outside W_l';

c) the normal bundle to W_l' in W_l is trivial.

We will consider the case when there is only 1 critical point on the level d_{l+1}. The general case differs only in the notation. Then the Weinstein domain W_{l+1} can be obtained from W_l by attaching a handle H of index k with an isotropic embedding $h : S \to \partial W_l$ of the core $(k-1)$-dimensional sphere $S \subset H$ with a saturation homomorphism $\Phi : TS \otimes \mathbb{C} \oplus \mathbb{C}^{n-k} \to \xi$, where ξ denotes the contact structure on the boundary of the Weinstein domain $(W_l, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$. According to Lemma 6.15 we can adjust the attaching map via an isotropic isotopy (which is the same as via ambient contact isotopy) to ensure that $h(S) \subset \partial W_l'$ and that the saturation Φ restricted to $E_1 = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k-1} \subset E = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k}$ is a saturation of the homomorphism $dh : S \to \xi'$, where $\xi' = \xi \cap (\partial W_l')$ is the induced contact structure on $\partial W_l'$. Then using Lemma 6.13 we can simultaneously attach index k handles to W_l and to W_l'. The resulted Weinstein structure on $(W_l \underset{h,\Phi}{\cup} H)$ coincides up to Weinstein homotopy with $(W_{l+1}, \omega, X_{l+1}, \phi)$ and we keep this notation for it. The Weinstein domain $W'_{l+1} = W'_l \underset{h,\Phi|_{E_1}}{\cup} H'$ is embedded in W_{l+1} in such a way that all the above properties a)-c) are satisfied. This gives a simultaneous handlebody description of Weinstein manifolds $(V'\omega')$ of dimension $2n-2$, and of $2n$-dimensional manifolds V, ω. Note that this handlebody decomposition of (V, ω) coincides with the decomposition of the stabilization $(V' \times \mathbb{R}^2, \omega' \oplus \omega_{\rm st})$, and hence, according to Propositions 6.9, 5.2 and Remark 6.11 the manifolds (V, ω) and $(V' \times \mathbb{R}^2, \omega' \oplus \omega_{\rm st})$ are symplectomorphic.

□

The following theorem is a slight modification of a result from [10].

Theorem 6.16. *Let $(V_1, \omega_1, X_1, \phi_1)$ and $(V_2, \omega_2, X_2, \phi_2)$ be two subcritical Weinsten manifolds. Suppose there exists a homotopy equivalence $h : V_1 \to V_2$ covered by a homomorphism $\Phi : TV_1 \to TV_2$ such that $\Phi^*\omega_2 = \omega_1$. Then h is homotopic to a symplectomorphism $f : (V_1, \omega_1) \to (V_2, \omega_2)$.*

6.4 Morse-Smale theory for Weinstein structures

Lemma 6.17. *Let (V, ω, X, ϕ) be a Weinstein structure. Let a be a regular value of ϕ, and p a critical point with $\phi(p) = b > a$. Suppose that all the trajectories of the vector field $-X$ emanating from p hit the level set $\Sigma_a = \{\phi = a\}$, i.e the intersection of the stable manifold of p with $\{\phi \geq a\}$ is a disc D with boundary $S = \partial D \subset \Sigma_a$.*

(i) Then for any $c \in (a, b]$ there is another Lyapunov Morse function $\widetilde{\phi}$ for X such that $\widetilde{\phi}(p) = b$, while all other critical values of $\widetilde{\phi}$ and ϕ coincide.

(ii) Given any contact isotopy $h_t : \Sigma_a \to \Sigma_a, t \in [0, 1]$, there is family of Weinstein structures (ω_t, X_t, ϕ) such that the stable manifold of the point p for X_t intersects Σ_a along $h_t(S)$, $t \in [0, 1]$.

(iii) Let q be another critical point of index $\operatorname{ind} q = \operatorname{ind} p - 1$ such that $\phi(q) = c < a$ and the intersection of the unstable manifold of p with $\{\phi \leq a\}$ is a disc Δ with boundary $\Sigma = \partial\Delta \subset \Sigma_a$. Suppose that S and Σ intersect transversely at 1 point. Then (V, ω) admits a Weinstein structure $(\widetilde{X}, \widetilde{\phi})$ such that $\operatorname{Crit}(\widetilde{\phi}) = \operatorname{Crit}(\phi) \setminus \{p, q\}$, where we denote by $\operatorname{Crit}(\widetilde{\phi})$ and $\operatorname{Crit}(\phi)$ the sets of critical points of the functions $\widetilde{\phi}$ and ϕ.

Chapter 7

Shapes for J-convex hypersurfaces

7.1 J-convex surroundings and extensions

For a closed subset A of a complex manifold (V, J), consider the following two problems.

Surrounding problem. Does A possess arbitrarily small neighbourhoods with smooth J-convex boundary?

Extension problem. Given a bounded J-convex function ϕ on a neighbourhood of A, does A possess arbitrarily small neighbourhoods U with smooth J-convex functions ψ such that $\psi = \phi$ near A and ∂U is a regular level set of ψ?

Lemma 7.1. *Let A be a closed subset of a complex manifold (V, J).*

(a) If the surrounding problem is solvable for A, then the extension problem is solvable for any bounded J-convex function ϕ near A.

(b) If (V, J) is Stein and the surrounding problem is solvable for a compact set A, then A possesses arbitrarily small neighbourhoods that are Stein. In particular, if (V, J) is Stein and ∂A is smooth and J-convex, then the interior of A is Stein.

Proof. (a) By hypothesis, A possesses arbitrarily small neighbourhoods U with smooth J-convex boundary. Let $c < \inf_A \phi$ and $C > \sup_A \phi$. By Lemma 1.3, there exists a J-convex surjective function $\tilde{\phi} : W \to [c, C+1]$ on a neighbourhood W of ∂U such that $\partial U = \tilde{\phi}^{-1}(C)$ is a regular level set. A smoothing of $\max(\phi, \tilde{\phi})$ is the desired function ψ.

(b) Pick any J-convex function $\phi : V \to \mathbb{R}$ and apply (a). $\square$

The preceding lemma reduces the extension problem to the surrounding problem. The same is true for certain refined extension problems considered below, although the reduction will not be quite as easy.

In Section 1.5 we have seen that the surrounding problem is solvable for

- totally real submanifolds;
- properly embedded complex hypersurfaces with negative normal bundle.

The first theorem of this chapter solves the surrounding problem for totally real balls suitably attached to J-convex domains. For a hypersurface Σ in an almost complex manifold (V, J), we say that a submanifold L with boundary $\partial L \subset \Sigma$ is attached *J-orthogonally* to Σ along ∂L if, for each point $p \in \partial L$, $JT_pL \subset T_p\Sigma$ and $T_pL \not\subset T_p\Sigma$. The first condition implies that ∂L is an integral submanifold for the maximal complex tangency ξ on Σ. If Σ is J-convex and $\dim_{\mathbb{R}} L = \dim_{\mathbb{C}} V = n$, then the second condition $T_pL \not\subset T_p\Sigma$ follows from the first one because integral submanifolds of the contact structure ξ have dimension at most $n-1$.

Theorem 7.2. *Let (V, J) be a complex manifold of complex dimension n and $W \subset V$ a compact domain with smooth J-convex boundary ∂W. Let $\Delta \subset V \setminus \mathrm{int}W$ be a real analytically embedded totally real k-ball attached J-orthogonally to ∂W along $\partial\Delta$. Let $V' \subset V$ be an open neighbourhood of $W \cup \Delta$. Then $W \cup \Delta$ has a compact neighbourhood $W' \subset V'$ with smooth J-convex boundary.*

Moreover, if $k < n$ and $f(D^k \times D^{n-k}) \subset V \setminus \mathrm{int}W$ is a real analytic and totally real embedding extending $\Delta = f(D^k \times \{0\})$, attached J-orthogonally to ∂W along $f(\partial D^k \times D^{n-k})$, then W' can be chosen such that $\partial W'$ intersects $f(D^k \times D^{n-k})$ J-orthogonally.

Corollary 7.3. *Let (V, J) be a complex manifold and $W \subset V$ a compact domain with smooth J-convex boundary ∂W. Let $L \subset V \setminus \mathrm{int}W$ be a real analytic and totally real compact submanifold attached J-orthogonally to ∂W along ∂L. Then $W \cup L$ has arbitrarily small neighborhoods with smooth J-convex boundary.*

Proof. Let $U \subset V$ be a given open neighbourhood of $W \cup L$. Pick a real analytic Morse function $\phi : L \to \mathbb{R}$ with regular level set $\partial L = \phi^{-1}(0)$ and critical points p_i of values $0 < \phi(p_1) < \cdots < \phi(p_m)$ and Morse indices k_i. Consider the gradient flow of ϕ with respect to some real analytic metric. (Such a metric exists by the results of Chapter 8.) The stable manifold $D^-(p_1)$ of p_1 is a real analytically embedded totally real k_1-ball attached J-orthogonally to ∂W. By Theorem 7.2, we find a compact neighbourhood $W_1 \subset U$ of $W \cup D^-(p_1)$ with smooth J-convex boundary. Moreover, due to the last statement in Theorem 7.2, we may assume that L intersects ∂W_1 J-orthogonally. In particular, $D^-(p_2)$ is attached J-orthogonally to W_1. Now continue by induction over the critical points. □

The preceding corollary extends to totally real immersions. We say that two totally real submanifolds L_1, L_2 of the same dimension in an almost complex manifold (V, J) *intersect J-orthogonally* at p if $JT_pL_1 = T_pL_2$.

Corollary 7.4. *Let (V, J) be a complex manifold and $W \subset V$ a compact domain with smooth J-convex boundary ∂W. Let $L \subset V \setminus \mathrm{int} W$ be a real analytic and totally real immersion of a compact manifold, with finitely many J-orthogonal self-intersections away from ∂L and attached J-orthogonally to ∂W along ∂L. Then $W \cup L$ has arbitrarily small neighborhoods with smooth J-convex boundary.*

Proof. Let $U \subset V$ be a given open neighbourhood of $W \cup L$. Let L_1, L_2 be the two local branches of L at a self-intersection point p. By J-orthogonality of the intersection, there exist local holomorphic coordinates in which $L_1 \subset \mathbb{R}^n$ and $L_2 \subset i\mathbb{R}^n$. Let $B(p) \subset U$ be the image in V of a small ball around the origin in $\mathbb{C}^n$. The boundary $\partial B(p)$ is J-convex and intersects L_1 and L_2 J-orthogonally. Construct such balls around all self-intersection points $p_1, \dots, p_m$, disjoint from each other and from ∂W. Then $W' := W \cup B(p_1) \cup \cdots \cup B(p_m) \subset U$ has J-convex boundary, to which the totally real submanifold $L \setminus \mathrm{int} W'$ is attached J-orthogonally. Hence the result follows from Corollary 7.3. □

In particular, for $W = \emptyset$ we obtain

Corollary 7.5. *Let (V, J) be a complex manifold and $L \subset V$ a real analytic and totally real immersion of a compact manifold with finitely many J-orthogonal self-intersections. Then L has arbitrarily small neighborhoods with smooth J-convex boundary.*

Remark 7.6. An alternative proof of the last corollary combines surroundings of totally real embeddings (Proposition 1.9) with the surroundings near the double points provided by Lemma 7.14 below.

The second theorem of this chapter allows us to extend J-convex functions over handles with control over the critical points.

Theorem 7.7. *Let (V, J) be a complex manifold of complex dimension n and $W \subset V$ a compact domain. Let $\Delta \subset V \setminus \mathrm{int} W$ be a real analytically embedded totally real k-ball attached J-orthogonally to ∂W along $\partial\Delta$. Let $\phi : W \to \mathbb{R}$ be a J-convex function with regular level set $\partial W = \phi^{-1}(a)$ which is extended to a function on Δ such that $\phi > a$ on $\mathrm{int}\Delta$. Then given any open neighbourhood $\tilde{V} \subset V$ of $W \cup \Delta$ and $b > \max_\Delta \phi$, there exists a compact neighbourhood $\tilde{W} \subset \tilde{V}$ of $W \cup \Delta$ and a J-convex function $\psi : \tilde{W} \to \mathbb{R}$ with the following properties:*

(a) $\psi = \phi$ on $W' := \{\phi \le a'\}$ for some $a' < a$;

(b) $\psi^{-1}(b)$ is a regular level set that coincides with $\phi^{-1}(a)$ outside a neighbourhood $U \subset V \setminus W'$ of $\partial\Delta$;

(c) $\psi = f \circ \phi$ on $W \setminus U$ for a smooth function $f : \mathbb{R} \to \mathbb{R}$;

(d) there exists an isotopy $h_t : \Delta' \to \Delta'$ (on an extension Δ' of Δ up to $\partial W'$) such that $h_t = 1\!\!1$ on $\Delta' \setminus U$, $h_0 = 1\!\!1$, and $h_1^* \phi = \psi$;

(e) the critical points of ψ agree with the critical points of $\phi|_\Delta$ and have positive definite Hessian transversely to Δ.

The proof of Theorems 7.2 and 7.7 will occupy the remainder of this chapter. It relies on the study of shapes for J-convex hypersurfaces.

7.2 Shapes for J-convex hypersurfaces

Consider the map

$$\pi : \mathbb{C}^n \to \mathbb{R}^2, \qquad z \mapsto (r, R) := (|x|, |y|)$$

for $z = x + iy$, $x, y \in \mathbb{R}^n$. The image of the map π is the quadrant

$$Q := \{(r, R) \mid r, R \geq 0\} \subset \mathbb{R}^2.$$

A curve $C \subset Q$ defines a hypersurface $\Sigma_C := \pi^{-1}(C)$ in $\mathbb{C}^n$. We call C the *shape* of Σ_C. Our goal in this section is to determine conditions on C which guarantee J-convexity of Σ_C.

As a preliminary, let us compute the second fundamental form of a surface of revolution. Consider $\mathbb{R}^k \oplus \mathbb{R}^l$ with coordinates (x, y) and $\mathbb{R}^k \oplus \mathbb{R}$ with coordinates $(x, R = |y|)$. To a function $\Phi : \mathbb{R}^k \oplus \mathbb{R} \to \mathbb{R}$ we associate the *surface of revolution*

$$\Sigma_\Phi := \{(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^l \mid \Phi(x, |y|) = 0\}.$$

We coorient Σ_Φ by the gradient $\nabla\Phi$ of Φ (with respect to all variables). Denote by $\Phi_R = \frac{\partial \Phi}{\partial R}$ the partial derivative.

Lemma 7.8. *At every $z = (x, y) \in \Sigma_\Phi$ the splitting*

$$T_z\Sigma_\Phi = \Big(T_z\Sigma_\Phi \cap (\mathbb{R}^k \oplus \mathbb{R}y)\Big) \oplus \Big(T_z\Sigma_\Phi \cap (\mathbb{R}^k \oplus \mathbb{R}y)^\perp\Big)$$

is orthogonal with respect to the second fundamental form II. The second subspace is an eigenspace of II with eigenvalue $\Phi_R/|\nabla\Phi|R$.

Proof. The unit normal vector to Σ_Φ at $z = (x, y)$ is

$$\nu(z) = \frac{1}{|\nabla\Phi|}(\nabla_x\Phi, \frac{\Phi_R}{R}y),$$

where $\nabla_x\Phi$ denotes the gradient with respect to the x-variables. For $Y \perp y$ we get

$$D\nu(z) \cdot (0, Y) = \frac{1}{|\nabla\Phi|}(0, \frac{\Phi_R}{R}Y) + \mu\nu$$

for some $\mu \in \mathbb{R}$. From $\langle \nu(z), D\nu(z) \cdot (0, Y)\rangle = 0$ we deduce $\mu = 0$, so $T_z\Sigma_\Phi \cap (\mathbb{R}^k \oplus \mathbb{R}y)^\perp$ is an eigenspace of II with eigenvalue $\Phi_R/|\nabla\Phi|R$. From this it follows that

$$II\Big((0, Y), (X, \lambda y)\Big) = \langle D\nu \cdot (0, Y), (X, \lambda y)\rangle = 0$$

for $(X, \lambda y) \in T_z\Sigma_\Phi \cap (\mathbb{R}^k \oplus \mathbb{R}y)$. □

Reduction to the case $n = 2$. Now let $C \subset Q$ be a curve. At a point $z = x + iy \in \Sigma_C$ consider the subspace $\Lambda_{xy} \subset \mathbb{R}^n$ generated by the vectors $x, y \in \mathbb{R}^n$ and its complexification

$$\Lambda_{xy}^{\mathbb{C}} := \Lambda_{xy} + i\Lambda_{xy}.$$

Let $\Lambda_\perp$ be the orthogonal complement of Λ_{xy} in $\mathbb{R}^n$ and $\Lambda_\perp^{\mathbb{C}}$ its complexification (which is the orthogonal complement of $\Lambda_{xy}^{\mathbb{C}}$ in $\mathbb{C}^n$). Note that $\Lambda_\perp^{\mathbb{C}}$ is contained in $T_z\Sigma_C$ and thus in the maximal complex subspace ξ_z. So the maximal complex subspace splits into the orthogonal sum (with respect to the metric)

$$\xi_z = \tilde{\Lambda} \oplus \Lambda_\perp^{\mathbb{C}} = \tilde{\Lambda} \oplus \Lambda_\perp \oplus i\Lambda_\perp,$$

where $\tilde{\Lambda} = \xi_z \cap \Lambda_{xy}^{\mathbb{C}}$. We claim that this splitting is orthogonal with respect to the second fundamental form II, and $\Lambda_\perp$ and $i\Lambda_\perp$ are eigenspaces with eigenvalues $\Phi_r/|\nabla\Phi|r$ and $\Phi_R/|\nabla\Phi|R$, respectively.

Indeed, Σ_C can be viewed as a surface of revolution in two ways, either rotating in the x- or the y-variables. So by Lemma 7.8, the splittings

$$\Big(\xi_z \cap (\mathbb{R}x \oplus i\mathbb{R}^n)\Big) \oplus \Big(\xi_z \cap (\mathbb{R}x \oplus i\mathbb{R}^n)^\perp\Big),$$
$$\Big(\xi_z \cap (\mathbb{R}^n \oplus i\mathbb{R}y)\Big) \oplus \Big(\xi_z \cap (\mathbb{R}^n \oplus i\mathbb{R}y)^\perp\Big)$$

are both orthogonal with respect to II and the right-hand spaces are eigenspaces. In particular, $\Lambda_\perp = \xi_z \cap (\mathbb{R}x \oplus i\mathbb{R}^n)^\perp$ and $i\Lambda_\perp = \xi_z \cap (\mathbb{R}^n \oplus i\mathbb{R}y)^\perp$ are eigenspaces orthogonal to each other with eigenvalues $\Phi_r/|\nabla\Phi|r$ and $\Phi_R/|\nabla\Phi|R$. Since $\Lambda_{xy}^{\mathbb{C}}$ is the orthogonal complement of $\Lambda_\perp \oplus i\Lambda_\perp$ in $\mathbb{C}^n$, the claim follows.

Now suppose that C is given near the point $\pi(z)$ by the equation $R = \phi(r)$, and the curve is cooriented by the gradient of the function $\Phi(r, R) = \phi(r) - R$. Since $|\nabla\Phi| = \sqrt{\Phi_r^2 + \Phi_R^2} = \sqrt{1 + \phi'(r)^2}$, the eigenvalues λ_r on $\Lambda_\perp$ and λ_R on $i\Lambda_\perp$ equal

$$\lambda_r = \frac{\Phi_r}{|\nabla\Phi|r} = \frac{\phi'(r)}{r\sqrt{1 + \phi'(r)^2}},$$
$$\lambda_R = \frac{\Phi_R}{|\nabla\Phi|R} = -\frac{1}{\phi(r)\sqrt{1 + \phi'(r)^2}}.$$

Hence by Proposition 1.8, the restriction of the Levi form L_{Σ_C} to $\Lambda_\perp^{\mathbb{C}}$ is given by

$$L_{\Sigma_C}(X, X) = \frac{1}{2\sqrt{1 + \phi'(r)^2}}\Big(\frac{\phi'(r)}{r} - \frac{1}{\phi(r)}\Big)|X|^2.$$

Hence we have proved

Lemma 7.9. *Let Σ_C be the hypersurface given by the curve $C = \{\phi(r) - R = 0\}$, cooriented by the gradient of $\phi(r) - R$. Then the restriction of the Levi form L_{Σ_C} to $\Lambda_\perp^{\mathbb{C}}$ is positive definite if and only if*

$$\mathcal{L}^\perp(\phi) := \frac{\phi'(r)}{r} - \frac{1}{\phi(r)} > 0.$$

In particular, if $\phi'(r) \leq 0$ the restriction is always negative definite.

Lemma 7.9 reduces the question about i-convexity of Σ_C to positivity of $\mathcal{L}^\perp(\phi)$ and the corresponding question about the intersection $\Sigma_C \cap \Lambda^{\mathbb{C}}_{xy}$. When $\dim_{\mathbb{C}} \Lambda^{\mathbb{C}}_{xy} = 1$, this intersection is a curve which is trivially J-convex, hence Σ_C is J-convex if and only if $\mathcal{L}^\perp(\phi) > 0$. The remaining case $\dim_{\mathbb{C}} \Lambda^{\mathbb{C}}_{xy} = 2$ just means that we have reduced the original question to the case $n = 2$, which we will now consider.

The case $n = 2$. We denote complex coordinates in $\mathbb{C}^2$ by $z = (\zeta, w)$ with $\zeta = s + it$, $w = u + iv$. The hypersurface $\Sigma_C \subset \mathbb{C}^2$ is given by the equation

$$\sqrt{t^2 + v^2} = R = \phi(r) = \phi(\sqrt{s^2 + u^2}).$$

We want to express the Levi form $\mathcal{L}$ at a point $z \in \Sigma_C$ in terms of ϕ. Suppose that $r, R > 0$ at the point z. After a unitary transformation

$$\zeta \mapsto \zeta \cos\alpha + w \sin\alpha, \qquad w \mapsto -\zeta \sin\alpha + w \cos\alpha$$

which leaves Σ_C invariant we may assume $t = 0$ and $v > 0$. Then near z we can solve the equation $R = \phi(r)$ for v,

$$v = \sqrt{\phi(\sqrt{s^2 + u^2})^2 - t^2} =: \psi(s, t, u).$$

According to Lemma 1.15, the Levi form of the hypersurface $\Sigma_C = \{v = \psi(s, t, u)\}$ is given by

$$\begin{aligned}\mathcal{L}(\psi) = {} & (\psi_{ss} + \psi_{tt})(1 + \psi_u^2) + \psi_{uu}(\psi_s^2 + \psi_t^2) \\ & + 2\psi_{su}(\psi_t - \psi_u\psi_s) - 2\psi_{tu}(\psi_s + \psi_u\psi_t).\end{aligned} \tag{7.1}$$

Note that at the point z we have $t = 0$ and $\psi(s, 0, u) = \phi(r) = \phi(\sqrt{u^2 + s^2})$. Using this, compute the derivatives at z,

$$\psi_s = \frac{\phi' s}{r}, \qquad \psi_{ss} = \frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3}, \qquad \psi_u = \frac{\phi' u}{r}, \qquad \psi_{uu} = \frac{\phi'' u^2}{r^2} + \frac{\phi' s^2}{r^3},$$
$$\psi_{su} = \frac{\phi'' su}{r^2} - \frac{\phi' su}{r^3}, \qquad \psi_t = 0, \qquad \psi_{tt} = -\frac{1}{\phi}, \qquad \psi_{tu} = 0.$$

Inserting this in equation (7.1), we obtain

$$\begin{aligned}\mathcal{L}(\psi) &= \Big(\frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3} - \frac{1}{\phi}\Big)\Big(1 + \frac{\phi'^2 u^2}{r^2}\Big) \\ &\quad + \Big(\frac{\phi'' u^2}{r^2} + \frac{\phi' s^2}{r^3}\Big)\frac{\phi'^2 s^2}{r^2} - 2\Big(\frac{\phi'' su}{r^2} - \frac{\phi' su}{r^3}\Big)\frac{\phi'^2 su}{r^2} \\ &= \frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3} + \frac{\phi'^3}{r} - \frac{1}{\phi}\Big(1 + \frac{\phi'^2 u^2}{r^2}\Big).\end{aligned}$$

We say that the curve C is *cooriented from above* if it is cooriented by the gradient of the function $\phi(r) - R$. Equivalently (since $t = 0$ at z), the hypersurface Σ_C is cooriented by the gradient of $\sqrt{\phi(\sqrt{s^2+u^2})^2 - t^2 - v}$, which is the coorientation we have chosen above. The opposite coorientation will be called *coorientation from below.* Lemma 7.9 and the preceding discussion yield the following criteria for i-convexity of Σ_C.

Proposition 7.10. *The hypersurface $\Sigma_C = \{R = \phi(r)\}$ is i-convex cooriented from above at $r > 0$ if and only if ϕ satisfies the following two conditions:*

$$\mathcal{L}^\perp(\phi) := \frac{\phi'(r)}{r} - \frac{1}{\phi(r)} > 0, \tag{7.2}$$

$$\mathcal{L}^2(\phi) := \frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3} + \frac{\phi'^3}{r} - \frac{1}{\phi}\Big(1 + \frac{\phi'^2 u^2}{r^2}\Big) > 0 \tag{7.3}$$

for all (s, u) with $s^2 + u^2 = r^2$. It is i-convex cooriented from below if and only if the reverse inequalities hold.

The following corollary gives some useful sufficient conditions for i-convexity.

Corollary 7.11. *(a) If $\phi > 0$, $\phi' > 0$, $\phi'' \le 0$ and*

$$\phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi}(1 + \phi'^2) > 0, \tag{7.4}$$

then Σ_C is i-convex cooriented from above.

(b) If $\phi > 0$, $\phi' \le 0$, $\phi'' \ge 0$ and

$$\phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi} < 0,$$

then Σ_C is i-convex cooriented from below.

Proof. (a) If $\phi' > 0$ and $\phi'' \le 0$ we get

$$\mathcal{L}^2(\phi) \ge \phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi}(1 + \phi'^2).$$

So positivity of the right hand side implies condition (7.3). Condition (7.2) is also a consequence of $\phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi}(1 + \phi'^2) > 0$.

(b) If $\phi' \le 0$ and $\phi'' \ge 0$ we get

$$\mathcal{L}^2(\phi) \le \phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi}.$$

So negativity of the right hand side implies the reverse inequality (7.3). The reverse inequality (7.2) is automatically satisfied. □

Remark 7.12. Note that if ϕ satisfies one of the inequalities (7.2), (7.3) and (7.4), then $\phi + c$ satisfies the same inequality for any constant $c \ge 0$. Thus i-convexity from above is preserved under upwards shifting.

7.3 Construction of special shapes

We will now construct special of shape functions satisfying the criteria of Corollary 7.11.

Proposition 7.13. *For all $\gamma, \epsilon > 0$ there exist $\delta > 0$, $K > 1$ with $K\delta < \gamma$, and a positive function $\phi : [\delta, K\delta] \to \mathbb{R}_+$ such that*

(a) $\phi'(\delta) = +\infty$, $\phi(\delta) > 1 - \gamma$;

(b) ϕ is strictly increasing and concave;

(c) at the point $K\delta$ the value and the first derivative of ϕ coincide with those of the function $r \mapsto 1 + \epsilon r$;

(d) ϕ satisfies the hypotheses of Corollary 7.11 (a), so $R = \phi(r)$ is the shape of an i-convex hypersurface cooriented from above.

Proof. One solution of the differential inequality in Corollary 7.11 (a) with the desired properties has been constructed in [9]. The following simplified proof was pointed out to us by M. Struwe. We will find the function ϕ as a solution of the differential equation

$$\phi'' + \frac{\phi'^3}{2r} = 0.$$

It will turn out that with an appropriate choice of the two free constants, ϕ then satisfies the conditions (a)-(d).

The differential equation is equivalent to

$$\left(\frac{1}{\phi'^2}\right)' = -\frac{2\phi''}{\phi'^3} = \frac{1}{r},$$

thus $1/\phi'^2 = \ln(r/\delta$ for some constant $\delta > 0$, or equivalently, $\phi'(r) = 1/\sqrt{\ln(r/\delta}$. By integration, this yields a solution ϕ for $r \geq \delta$ which is strictly increasing and concave and satisfies $\phi'(\delta) = \infty$. Define the constant $K > 1$ by the condition $\phi'(K\delta) = \epsilon$. Solving for K yields $K = e^{1/\epsilon^2}$, so K depends only on ϵ.

Note that $\int_\delta^{K\delta} \phi'(r)dr = \delta \int_1^K \frac{du}{\sqrt{\ln u}}$. Set

$$K_1 := \int_1^K \frac{du}{\sqrt{\ln u}} < \infty$$

and choose δ so small that $K\delta < \gamma$ and $K_1\delta < \gamma$. We fix the remaining free constant in ϕ by defining

$$\phi(r) := 1 + \epsilon K\delta - \int_r^{K\delta} \frac{dr}{\sqrt{\ln(r/\delta)}}$$

for $r \geq \delta$. The value and the derivative of ϕ at the point $K\delta$ coincide with those of the the function $1 + \epsilon r$. We also have

$$\phi(\delta) = 1 + \epsilon K\delta - K_1\delta > 1 - \gamma.$$

So it remains only to check the inequality of Corollary 7.11 (a), which in view of the differential equation for ϕ reduces to

$$\frac{\phi'^3}{2r} - \frac{1}{\phi}(1+\phi'^2) > 0.$$

We already know that $\phi(r) > 1-\epsilon$. Assuming $\epsilon \le 1/2$, we get

$$\begin{aligned}\frac{\phi'^3}{2r} - \frac{1}{\phi}(1+\phi'^2) > \frac{\phi'^3}{2r} - 2(1+\phi'^2) &= \frac{1}{2}\Big(\frac{1}{r\big(\ln(r/\delta)\big)^{3/2}} - 4 - \frac{4}{\ln(r/\delta)}\Big)\\ &= \frac{1}{2\ln(r/\delta)}\Big(\frac{1}{r\sqrt{\ln(r/\delta)}} - 4\ln(r/\delta) - 4\Big).\end{aligned}$$

The function in brackets is decreasing in r. So its minimum is achieved for $r = K\delta$ and has the value

$$\frac{1}{\delta K\sqrt{\ln K}} - 4\ln K - 4,$$

which is positive if δ is sufficiently small. □

Lemma 7.14.

7.4 Families of special shapes

For numbers $\lambda, a, b, c, d \ge 0$ consider the following functions:

$S_\lambda(r) = \sqrt{\lambda^2 + ar^2}$ (standard function),

$Q_\lambda(r) = \lambda + br + cr^2/2\lambda$ (quadratic function),

$L_\lambda(r) = \lambda + dr$ (linear function).

Let us first determine in which ranges they satisfy the inequalities (7.2) and (7.3).

Lemma 7.15. *(a) The function $S_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda \ge 0$, $a > 1$ and $r > 0$.*

(b) The function $Q_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda > 0$, $b \ge 0$, $c > 1$ and $r > 0$.

(c) The function $Q_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda > 0$, $b = 4 - c$, $0 \le c \le 4$ and $0 < r \le 2\lambda$.

(d) The function $L_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda \ge 0$, $d > 1$ and $r > 0$.

(e) The function $L_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda > 0$, $d > 0$, $r > 0$ and $r(1-d^4) < \lambda d^3$.

Proof. First note that we only need to prove the statements for $\lambda = 1$. This follows from the fact that if a function ϕ is the shape of an i-convex hypersurface at $r > 0$, then the function $\phi_\lambda(r) := \lambda\phi(r/\lambda)$ is the shape of an i-convex hypersurface at λr for each $\lambda > 0$. This fact can be seen by applying the biholomorphism $z \mapsto \lambda z$ on $\mathbb{C}^n$, or from Proposition 7.10 as follows: The function ϕ_λ has derivatives $\phi_\lambda(\lambda r) = \lambda\phi(r)$, $\phi'_\lambda(\lambda r) = \phi'(r)$, $\phi''_\lambda(\lambda r) = \phi(r)/\lambda$, and the replacement $r \mapsto \lambda r$, $\phi \mapsto \lambda\phi$, $\phi' \mapsto \phi'$, $\phi'' \mapsto \phi''/\lambda$ leaves both conditions in Proposition 7.10 unchanged. Set $S := S_1$, $Q := Q_1$, $L := L_1$. We denote by $\sim$ equality up to multiplication by a positive factor.

(a) This holds because $R = S(r)$ describes a level set of the i-convex function $\phi(r, R) = ar^2 - R^2$ for $a > 1$.

(b) Condition (7.2) follows from

$$Q'(r)Q(r) - r = (b + cr)(1 + br + \frac{cr^2}{2}) - r \geq b + cr - r = b + (c-1)r > 0,$$

and condition (7.3) from

$$\begin{aligned}
\mathcal{L}^2(Q) &\geq \frac{c(r^2 - u^2)}{r^2} + \frac{(b+cr)u^2}{r^3} + \frac{(b+cr)^3}{r} - 1 - \frac{(b+cr)^2u^2}{r^2} \\
&\sim cr(r^2 - u^2) + (b+cr)u^2 + r^2(b+cr)^3 - r^3 - r(b+cr)^2u^2 \\
&= (c-1)r^3 + bu^2 + r^2(b+cr)^3 - ru^2(b+cr) \\
&\geq (c-1)r^3 + +r^2(b+cr)^3 - r^3(b+cr) \\
&= (c-1)r^3 + r^2(b+cr)^2\big(b + (c-1)r\big) > 0.
\end{aligned}$$

(c) Condition (7.2) follows as in (b) from

$$Q'(r)Q(r) - r \geq b + cr - r = 4 - c(1-r) - r \geq 4 - 4(1-r) = 4r - r > 0.$$

For condition (7.3) it suffices, by (b), to show that

$$\begin{aligned}
A &:= (c-1)r + (b+cr)^2\big(b + (c-1)r\big) \\
&= (c-1)r + \big(4 - c(1-r)\big)^2\big(4 - c(1-r) - r\big) > 0.
\end{aligned}$$

For $c > 1$ this follows from (b). For $c \leq 1$ we have $4 - c(1-r) \geq 3$ and $4 - c(1-r) - r \geq 3 - r$, hence

$$A \geq -r + 9(3-r) = 27 - 10r > 0$$

for $r \leq 2$.

(d) Condition (7.2) follows from

$$L'(r)L(r) - r = d(1 + dr) - r = d + (d^2 - 1)r > 0,$$

and condition (7.3) from

$$\begin{aligned}\mathcal{L}^2(L) &= \frac{du^2}{r^3} + \frac{d^3}{r} - \frac{1}{1+dr}\Big(1 + \frac{d^2u^2}{r^2}\Big)\\ &\sim (1+dr)du^2 + d^3r^2(1+dr) - r^3 - d^2ru^2\\ &= du^2 + d^3r^2 + (d^4-1)r^3 > 0.\end{aligned}$$

(e) Condition (7.2) follows from $r(1-d^2) < d^3/(1+d^2)$ via

$$L'(r)L(r) - r = d + (d^2-1)r \geq d - \frac{d^3}{1+d^2} = \frac{d}{1+d^2} > 0,$$

and condition (7.3) as in (d) from

$$\mathcal{L}^2(L) = du^2 + d^3r^2 + (d^4-1)r^3 \geq r^2\big(d^3 + (d^4-1)r\big) > 0.$$

□

Lemma 7.16. *(a) For $\lambda, c > 0$ and $d > b > 0$ the functions $Q_\lambda(r)$ and $L_\lambda(r)$ intersect at the point $r = 2(d-b)\lambda/c$.*

(b) For $\lambda > 0$ and $a > d^2 > 0$ the functions $L_\lambda(r)$ and $S_\lambda(r)$ intersect at the point $r = 2d\lambda/(a-d^2)$.

(c) For $\lambda, b > 0$, $a > c \geq 0$ and $2b^2(a+c)^2 < (a-c)^3$ the functions $S_\lambda(r)$ and $Q_\lambda(r)$ intersect at a point r satisfying $0 < r < 4b\lambda/(a-c)$.

Proof. (a) and (b) are simple computations, so we only prove (c). Again, by rescaling it suffices to consider the case $\lambda = 1$. First observe that for $x > 0$ and $\mu < 1$ we have $\sqrt{1+x} > 1 + \mu x/2$ provided that $1 + x > 1 + \mu x + \mu^2x^2/4$, or equivalently, $x < 4(1-\mu)/\mu^2$. Applying this to $x = ar^2$, we find that $S(r) > 1 + \mu ar^2/2$ provided that

$$r^2 < \frac{4(1-\mu)}{a\mu^2}. \tag{7.5}$$

Since $b > 0$, we have $S(r) < Q(r)$ for $r > 0$ very small. Hence if

$$1 + \frac{\mu ar^2}{2} = Q(r) = 1 + br + \frac{cr^2}{2}$$

for some $r > 0$ and $\mu < 1$ satisfying (7.5), then $S(r)$ and $Q(r)$ intersect in the interval $(0, r)$. Assuming $\mu a > c$, we solve the last equation for $r = 2b/(\mu a - c)$. Inequality (7.5) becomes

$$r^2 = \frac{4b^2}{(\mu a - c)^2} < \frac{4(1-\mu)}{a\mu^2},$$

or equivalently,

$$ab^2\mu^2 < (1-\mu)(\mu a - c)^2. \tag{7.6}$$

So if $\mu < 1$ satisfies $\mu a > c$ and (7.6), then $S(r)$ and $Q(r)$ intersect at a point r satisfying $0 < r < 2b/(\mu a - c)$.

Now pick $\mu := (a+c)/2a$. The hypothesis $a > c$ implies $\mu < 1$ and $\mu a = (a+c)/2 > c$. With $\mu a - c = (a-c)/2$ and $1 - \mu = (a-c)/2a$, inequality (7.6) becomes

$$ab^2 \left(\frac{a+c}{2a}\right)^2 < \frac{a-c}{2a}\left(\frac{a-c}{2}\right)^2,$$

or equivalently,

$$2b^2(a+c)^2 < (a-c)^3.$$

If this inequality holds, then $S(r)$ and $Q(r)$ intersect at a point r satisfying

$$0 < r < \frac{2b}{\mu a - c} = \frac{4b}{a-c}.$$

□

Now the numerical inequality $32(64+2)^2 < (64-2)^3$ implies

Corollary 7.17. *For $\lambda > 0$, $a = 64$, $c = 2$ and $0 < b \leq 4$, the functions $S_\lambda(r)$ and $Q_\lambda(r)$ are i-convex and intersect at a point r satisfying $0 < r < b\lambda/8$.*

We also have

Lemma 7.18. *For every $a > 1$ and $\gamma > 0$ there exists a $0 < d < \gamma$ and an i-convex shape $\phi(r)$ which agrees with $S(r) = \sqrt{1+ar^2}$ for $r \geq \gamma$ and with $L(r) = 1 + dr$ for r close to 0.*

Proof. Pick $1 < c < a$. Pick $0 < b < 1$ such that $2b^2(a+c)^2 < (a-c)^3$ and $4b < \gamma(a-c)$. By Lemma 7.16, the i-convex shapes $S(r)$ and $Q(r) = 1+br+cr^2/2$ intersect at a point $0 < r_2 < 4b/(a-c) < \gamma$. Now pick $b < d < 1$ such that $r_1 := 2(d-b)/c$ satisfies $r_1 < r_2$ and $r_1 < d^3/(1-d^4)$. By Lemma 7.16, the functions $Q(r)$ and $L(r)$ intersect at the point r_1, and by Lemma 7.15 the function $L(r)$ is i-convex for $r \leq r_1$. Now the desired function is a smoothing of the function which equals $L(r)$ for $r \leq r_1$, $Q(r)$ for $r_1 \leq r \leq r_2$ and $S(r)$ for $r \geq r_2$. □

Combining the preceding lemma with Proposition 7.13, we obtain

Corollary 7.19. *For every $a > 1$ and $\gamma > 0$ there exists a $0 < \delta < \gamma$ and an i-convex shape $\phi(r)$ which agrees with $S(r) = \sqrt{1+ar^2}$ for $r \geq \gamma$ and satisfies $\phi'(\delta) = +\infty$ and $\phi(\delta) > 1 - \gamma$.*

We need a refinement of Proposition 7.13. Consider again a solution of Struwe's differential equation

$$\phi'' + \frac{\phi'^3}{2r} = 0 \tag{7.7}$$

with $\phi' > 0$ and hence $\phi'' < 0$. Then the inequality in Corollary 7.11 (a) reduces to

$$\frac{\phi'^3}{2r} - \frac{1}{\phi}(1+\phi'^2) > 0. \tag{7.8}$$

Lemma 7.20. *For any $d, K, \delta, \lambda > 0$ satisfying $K \geq e^{4/d^2}$ and $4K\delta \leq (\ln K)^{-3/2}$ there exists a solution $\phi : [\lambda\delta, K\lambda\delta] \to \mathbb{R}$ of (7.7) with the following properties:*

(a) $\phi'(\lambda\delta) = +\infty$ and $\phi(\lambda\delta) \geq \lambda + d\lambda\delta$;

(b) $\phi(K\lambda\delta) = \lambda + dK\lambda\delta$ and $\phi'(K\lambda\delta) \leq d$;

(c) ϕ satisfies (7.8) and hence is the shape of an i-convex hypersurface cooriented from above.

Proof. Again it suffices to consider the case $\lambda = 1$; the general case then follows by replacing ϕ by $\lambda\phi(r/\lambda)$. The differential equation (7.7) is equivalent to

$$\Big(\frac{1}{\phi'^2}\Big)' = -\frac{2\phi''}{\phi'^3} = \frac{1}{r},$$

thus $1/\phi'^2 = \ln(r/\delta)$ for some constant $\delta > 0$, or equivalently, $\phi'(r) = 1/\sqrt{\ln(r/\delta)}$. By integration, this yields a solution ϕ for $r \geq \delta$ which is strictly increasing and concave and satisfies $\phi'(\delta) = +\infty$. Note that $\int_\delta^{K\delta} \phi'(r)dr = \delta K_1$ with

$$K_1 := \int_1^K \frac{du}{\sqrt{\ln u}} < \infty.$$

Fix the remaining free constant in ϕ by setting $\phi(K\delta) := 1 + dK\delta$, thus

$$\phi(\delta) = 1 + dK\delta - K_1\delta.$$

Estimating the logarithm on $[1, K]$ from below by the linear function with the same values at the endpoints,

$$\ln u \geq \frac{\ln K}{K-1}(u-1),$$

we obtain an upper estimate for K_1:

$$K_1 \leq \int_1^K \frac{du}{\sqrt{\frac{\ln K}{K-1}(u-1)}} = \sqrt{\frac{K-1}{\ln K}} \int_0^{K-1} \frac{du}{\sqrt{u}} = \frac{2(K-1)}{\sqrt{\ln K}}. \tag{7.9}$$

By hypothesis we have $\sqrt{\ln K} \geq 2/d$, hence $K_1 \leq d(K-1)$. This implies

$$\phi(\delta) \geq 1 + dK\delta - d(K-1)\delta = 1 + d\delta.$$

Concavity of ϕ implies $\phi(r) \geq 1 + dr$ for all $r \in [\delta, K\delta]$, and in particular $\phi'(K\delta) \leq d$. So it only remains to check inequality (7.8). Denoting by $\sim$

equality up to a positive factor, we compute

$$\begin{aligned}\frac{\phi'^3}{2r} - \frac{1}{\phi}(1+\phi'^2) &\geq \frac{\phi'^3}{2r} - \frac{1}{1+dr}(1+\phi'^2) \\ &\sim \frac{\phi'^3}{r}(1+dr) - 2 - 2\phi'^2 \\ &\sim \frac{1}{r} + d - 2\ln(r/\delta)^{3/2} - 2\ln(r/\delta)^{1/2}.\end{aligned}$$

The function on the right hand side is decreasing in r. So its minimum is achieved for $r = K\delta$ and has the value

$$\frac{1}{K\delta} - 2(\ln K)^{3/2} - 2(\ln K)^{1/2} > \frac{1}{K\delta} - 4(\ln K)^{3/2} \geq 0$$

by hypothesis. □

Lemma 7.21. *For any $\delta > 0$ and $d \geq 4$ there exists a solution $\phi : [\delta, 2\delta] \to \mathbb{R}$ of (7.7) with the following properties:*

(a) $\phi'(\delta) = +\infty$ and $\phi(\delta) \geq d\delta$;

(b) $\phi(2\delta) = 2d\delta$ and $\phi'(2\delta) \leq d$;

(c) ϕ satisfies (7.8) and hence is the shape of an i-convex hypersurface cooriented from above.

Proof. The proof is similar to the proof of Lemma 7.20. By rescaling, it suffices to consider the case $\delta = 1$. Define the solution ϕ by $\phi'(r) := 1/\sqrt{\ln r}$ and $\phi(2) := 2d$, thus

$$\phi(1) = 2d - \int_1^2 \frac{du}{\sqrt{\ln u}}.$$

Estimating the integral as in (7.9) and using $d \geq 4$, we find

$$\phi(1) \geq 2d - \frac{2}{\sqrt{\ln 2}} \geq d + 4 - \frac{2}{\sqrt{\ln 2}} \geq d,$$

since $\sqrt{\ln 2} \geq 1/2$. Concavity of ϕ implies $\phi(r) \geq dr$ for all $r \in [1, 2]$, and in particular $\phi'(2) \leq d$. So it only remains to check inequality (7.8). Denoting by $\sim$ equality up to a positive factor, we compute

$$\begin{aligned}\frac{\phi'^3}{2r} - \frac{1}{\phi}(1+\phi'^2) &\geq \frac{\phi'^3}{2r} - \frac{1}{dr}(1+\phi'^2) \\ &\sim d\phi'^3 - 2 - 2\phi'^2 \\ &\sim d - 2(\ln r)^{3/2} - 2(\ln r)^{1/2}.\end{aligned}$$

The function on the right hand side is decreasing in r. So its minimum is achieved for $r = 2$ and has the value

$$d - 2(\ln 2)^{3/2} - 2(\ln 2)^{1/2} > 4 - 2 - 2 = 0,$$

since $d \geq 4$ and $\sqrt{\ln 2} < 1$. □

Remark 7.22. For ϕ as in Lemma 7.21 and any constant $c \in \mathbb{R}$, the part of the function $\phi + c$ that lies above the linear function dr is i-convex. Indeed, the last part of the proof applied to $\phi + c$ estimates the quantity in inequality 7.8 by $d - 2(\ln r_1)^{3/2} - 2(\ln r_1)^{1/2}$, where r_1 is the larger intersection point of $\phi + c$ and dr. Since $r_1 \leq 2$, this is positive.

Extend the standard function to $\lambda < 0$ and $a > 1$ by

$$S_\lambda(r) := \sqrt{ar^2 - \lambda^2}, \qquad r \geq |\lambda|/\sqrt{a}.$$

Note that S_λ is the shape of an i-convex hypersurface because its graph is a level set of the i-convex function $\phi(r, R) = ar^2 - R^2$.

Proposition 7.23. *Let $L_\lambda(r) = \lambda + d_\lambda r$, $0 < r \leq \gamma$, $0 \leq \lambda \leq 1$, be an increasing smooth family of i-convex shape functions, where $\lambda \mapsto b_\lambda$ is decreasing with $d_0 = 8$ and $0 < d_1 \leq 1$. Then there exists a $0 < \delta < \gamma/4$ and a smooth family of increasing functions $\psi_\lambda : [\delta, \gamma] \to \mathbb{R}_+$, $-8\delta \leq \lambda \leq 1$, with the following properties:*

(a) $\psi_{-8\delta}(r) = \sqrt{64r^2 - 64\delta^2}$ for all $r \geq \delta$;

(b) $\psi_\lambda(r) = \sqrt{64r^2 - 64\delta^2}$ for $\delta \leq r \leq 2\delta$ and all λ;

(c) $\psi_\lambda(r) = \sqrt{64r^2 - \lambda^2}$ for $-8\delta \leq \lambda \leq 0$ and $r \geq \gamma/2$;

(d) $\psi_\lambda(r) = L_\lambda(r)$ for $0 \leq \lambda \leq 1$ and $r \geq \gamma/2$;

(e) each ψ_λ is the shape of an i-convex hypersurface cooriented from above.

Proof. (1) For each $\lambda \in (0, 1]$, set $K_\lambda := e^{4/d_\lambda^2}$. Pick a smooth family of $\delta_\lambda > 0$ such that $K_\lambda \delta_\lambda$ decreases with λ and

$$4K_\lambda\delta_\lambda \leq (\ln K_\lambda)^{-3/2}, \qquad \lambda K_\lambda \delta_\lambda < \gamma/4.$$

By Lemma 7.20, there exist i-convex solutions $\phi_\lambda : [\lambda\delta_\lambda, K_\lambda\lambda\delta_\lambda] \to \mathbb{R}$ of (7.7) satisfying

- $\phi'_\lambda(\lambda\delta_\lambda) = +\infty$ and $\phi_\lambda(\lambda\delta_\lambda) \geq \lambda + d_\lambda\lambda\delta_\lambda$;
- $\phi_\lambda(K_\lambda\lambda\delta_\lambda) = \lambda + d_\lambda K_\lambda\lambda\delta_\lambda$ and $\phi'_\lambda(K_\lambda\lambda\delta_\lambda) \leq d_\lambda$.

(2) From $d_0 = 8$ and $d_1 < 1$ we conclude $K_0 = e^{1/16} < 2$ and $K_1 \geq e^4 > 2$. Hence there exists a $\bar\lambda > 0$ with $K_{\bar\lambda} = 2$. Set $\bar\delta := \bar\lambda\delta_{\bar\lambda} < \gamma/4$. By Lemma 7.21 (with $d = 8$), there exists an i-convex solution $\bar\phi : [\bar\delta, 2\bar\delta] \to \mathbb{R}$ of (7.7) satisfying

- $\bar\phi'(\bar\delta) = +\infty$ and $\bar\phi(\bar\delta) \geq 8\bar\delta$;
- $\bar\phi(2\bar\delta) = 16\bar\delta$ and $\bar\phi'(2\bar\delta) \leq 8$.

By the remark following Corollary 7.11, the functions

$$\bar\phi_\lambda := \bar\phi(r) + L_\lambda(2\bar\delta) - L_0(2\bar\delta) \geq \bar\phi(r)$$

are i-convex for $0 \le \lambda \le \bar\lambda$ and $\bar\delta \le r \le 2\bar\delta$. Note that the functions $\phi_{\bar\lambda}$ and $\bar\phi_{\bar\lambda}$ have the same value at $r = 2\bar\delta$ and derivative ∞ at $r = \bar\delta$. Since they both solve the second order differential equation (7.7), they coincide on $[\bar\delta, 2\bar\delta]$. Thus the families constructed in (1) and (2) fit together to a continuous family $(\hat\phi_\lambda)_{\lambda\in[0,1]}$ with $\hat\phi_\lambda = \phi_\lambda : [\delta_\lambda, K_\lambda\delta_\lambda] \to \mathbb{R}_+$ for $\lambda \ge \bar\lambda$, and $\hat\phi_\lambda = \bar\phi_\lambda : [\bar\delta, 2\bar\delta] \to \mathbb{R}_+$ for $\lambda \le \bar\lambda$. Define $\tilde\phi_\lambda : [\bar\delta, \gamma] \to \mathbb{R}_+$ by

$$\tilde\phi_\lambda(r) := \begin{cases} \hat\phi_\lambda(r) & \text{for } r \le K_\lambda\delta_\lambda, \\ L_\lambda(r) & \text{for } r \ge K_\lambda\delta_\lambda. \end{cases}$$

After smoothing, the family $\tilde\phi_\lambda$ is i-convex and agrees with L_λ for $r \ge \gamma/2$.

(3) For $-8\bar\delta \le \tau \le 0$ consider the functions $\bar\phi_\tau := \bar\phi + \tau : [\delta, 2\delta] \to \mathbb{R}_+$. By the remark following Lemma 7.21, the portion of $\bar\phi_\tau$ above the linear function L_0 is i-convex. Thus for $0 < \delta < \bar\delta/2$ sufficiently small, the portion of $\bar\phi_\tau$ above the function $S_{-8\delta}$ is i-convex. For $-8\delta \le \lambda \le 0$ define $\tilde\phi_\lambda : [\bar\delta, \gamma] \to \mathbb{R}_+$ by

$$\tilde\phi_\lambda(r) := \begin{cases} \bar\phi(r) + S_\lambda(2\bar\delta) - S_0(2\bar\delta) & \text{for } r \le \bar\delta, \\ S_\lambda(r) & \text{for } r \ge 2\bar\delta. \end{cases}$$

After smoothing, the family $\tilde\phi_\lambda$ is i-convex for $-8\delta \le \lambda \le 1$ and agrees with L_λ for $r \ge \gamma/2$. Now define $\tilde\psi_\lambda : [\delta, \gamma] \to \mathbb{R}_+$ by

$$\tilde\phi_\lambda(r) := \begin{cases} S_{-8\delta(r)} & \text{for } r \le \bar\delta, \\ \tilde\phi(r) & \text{for } r \ge \bar\delta. \end{cases}$$

After smoothing, the family $\tilde\psi_\lambda$ is i-convex for $-8\delta \le \lambda \le 1$ and satisfies conditions (b-e).

Note that $\tilde\psi_{-8\delta} = \max(S_{-8\delta}, \bar\phi_{\bar\tau})$ for some $\bar\tau < 0$. By the discussion above, the functions $\max(S_{-8\delta}, \bar\phi_\tau)$ are i-convex for $-8\bar\delta \le \tau \le 0$. For δ sufficiently small, we have $\max(S_{-8\delta}, \bar\phi_{-8\bar\delta}) = S_{-8\delta}$. After rescaling in the parameter λ, this yields the desired family ψ_λ. □

7.5 J-convex surroundings and extensions on a handle

In this section we solve the surrounding and extension problems for the totally real core of a standard handle.

The following lemma extends shapes to the subcritical case.

Lemma 7.24. *For $k < n$ set $r := \sqrt{x_1^2 + \cdots + x_n^2 + y_{k+1}^2 + \cdots + y_n^2}$ and $R := \sqrt{y_1^2 + \cdots + y_k^2}$. Let $\phi(r)$ be a function satisfying the conditions of Corollary 7.11 (a). Then $\Sigma := \{R = \phi(r)\}$ is an i-convex hypersurface cooriented from above. Moreover, Σ intersects the subspace $i\mathbb{R}^n$ i-orthogonally.*

Proof. Set $\bar r := \sqrt{x_1^2+\cdots+x_n^2}$ and $\bar R := \sqrt{y_1^2+\cdots+y_n^2}$. By Corollary 7.11, the hypersurface $\bar\Sigma := \{\bar R = \phi(\bar r)\}$ is i-convex cooriented from above. Let $\bar\psi(\bar r,\bar R)$ be an increasing function of $\phi(\bar r)-\bar R$ which is i-convex on a neighbourhood of $\bar\Sigma$. The unitary group $U(n-k)$ acts on the second factor of $\mathbb{C}^n=\mathbb{C}^k\oplus\mathbb{C}^{n-k}$ and the functions $z\mapsto\bar\psi(gz)$, $g\in U(n-k)$, form a smooth family of i-convex functions. Therefore, by Lemma 2.8, the continuous function

$$\psi(z) := \max_{g\in U(n-k)}\bar\psi(gz)$$

is i-convex. Set $z' := (z_1,\dots,z_k)$ and $z'' := (z_{k+1},\dots,z_n)$. Since $\bar\phi$ is increasing, the function

$$g\mapsto\bar\phi\big(\sqrt{\mathrm{Re}\,(z')^2+\mathrm{Re}\,(gz'')^2}\big)-\sqrt{\mathrm{Im}\,(z')^2+\mathrm{Im}\,(gz'')^2}$$

for fixed (z',z'') is maximized iff $\mathrm{Im}\,(gz'')=0$, so we have $\psi(z)=\bar\psi(r,R)$. This implies that $\psi(z)=\bar\psi(r,R)$ is smooth and i-convex, hence its level set Σ is also i-convex.

The i-orthogonality of Σ to $i\mathbb{R}^n$ is clear from the definition. □

We use the notation of Section 6.1. Let r,R be as above. The following is the key result for the proof of the Extension Theorem 7.7.

Proposition 7.25. *Let H be a standard k-handle and $\phi:U\to\mathbb{R}$ an i-convex function on a neighbourhood of S such that $\phi|_S\equiv a$ and $d\phi=-2dR$ along S. Extend ϕ to a function $D\cup U\to\mathbb{R}$ such that $\phi>a$ on $\mathrm{int}D$. Let $\tilde U$ be a neighbourhood of S in U and $b>\max_D\phi$. Then there exists a neighbourhood $W\subset H$ of $\{\phi\le a\}\cup D$ and an i-convex function $\psi:W\to\mathbb{R}$ with the following properties:*

(a) $\psi=\phi$ on $\{\phi\le a'\}$ for some $a'<a$;

(b) $\psi^{-1}(b)$ is a regular level set that coincides with $\phi^{-1}(a)$ outside $\tilde U$;

(c) $\psi=f\circ\phi$ on $\{\phi\le a\}\setminus\tilde U$ for a smooth function $f:\mathbb{R}\to\mathbb{R}$;

(d) there exists an isotopy $h_t:D_{1+\varepsilon}\to D_{1+\varepsilon}$ such that $h_t=\mathbb{1}$ outside $D_{1+\varepsilon}\cap\tilde U$, $h_0=\mathbb{1}$, and $h_1^*\phi=\psi$;

(e) the critical points of ψ agree with the critical points of $\phi|_D$ and have positive definite Hessian transversely to D.

Proof. Fix $A>1$. By hypothesis, ϕ coincides together with its differential along $S=\{r=0,R=1\}$ with the i-convex function Ar^2-R^2+1+a. By Proposition 2.15, there exists an i-convex function $\tilde\phi:U\to\mathbb{R}$, C^1-close to ϕ, with $\tilde\phi=\phi$ outside $\tilde U$ and $\tilde\phi=Ar^2-R^2+1+a$ near S. Since $\tilde\phi$ and ϕ are C^1-close and have no critical points on (sufficiently small) $\tilde U$, there exists and isotopy $h_t:D_{1+\varepsilon}\to D_{1+\varepsilon}$ such that $h_t=\mathbb{1}$ outside $D_{1+\varepsilon}\cap\tilde U$, $h_0=\mathbb{1}$, and $h_1^*\phi=\tilde\phi$.

Extend $\tilde\phi$ to an open neighbourhood $\tilde W$ of $U\cup D$ by $\tilde\phi|_D+Ar^2$ outside U. This function will be i-convex for A sufficiently large. Choose $\tilde W$ so small that $\sup_{\tilde W}\tilde\phi<b$.

By construction, the level set $\tilde\phi^{-1}(a)$ agrees with the hypersurface $\{R = \sqrt{1+Ar^2}\}$ on $r \le \gamma$ for some $\gamma > 0$. By Corollary 7.19 and Lemma 7.24, there exists an i-convex hypersurface $\Sigma \subset \tilde W$ (given by $\{R = \varphi(r)\}$ for a suitable shape φ) which surrounds the disk D and coincides with $\tilde\phi^{-1}(a)$ along $r = \gamma$.

Choose a tubular neighborhood $\widetilde{\Sigma[-1,1]}$ of $\Sigma = \widetilde{\Sigma\{0\}}$ in $\tilde W$ such that the hypersurfaces $\Sigma_t := \widetilde{\Sigma\{t\}}$ are i-convex, and outside $\tilde U$ they coincide with level surfaces of the function ϕ. By Lemma 1.3, there exists an i-convex function $\zeta : \widetilde{\Sigma[-1,1]} \to \mathbb{R}$ with level sets Σ_t such that $\zeta|_{\Sigma_0} = b$ and $\zeta_{\Sigma_{-1}} = b' < \inf_{\tilde W} \tilde\phi$. Extend ζ to the domain bounded by Σ_{-1} as the constant b' and set

$$\tilde\psi := \max(\tilde\phi, \zeta)$$

on the domain $W := \{\zeta \le b\} \subset \tilde W$ bounded by Σ. Note that $\tilde\psi = \tilde\phi$ in the region $\{\zeta = b'\}$ bounded by Σ_{-1}, hence $\tilde\psi$ is strictly i-convex (although the constant function b' is not) and $\tilde\psi = \tilde\phi$ near D. In particular, the critical points of $\tilde\psi$ on $\{\zeta = b'\}$ agree with the critical points of $\phi|_D$ and have positive definite Hessian transversely to D. On the other hand, $\tilde\psi = \zeta$ near Σ, and in particular $\Sigma = \tilde\psi^{-1}(b)$.

Observe that on $W \setminus \tilde U$ we have $\tilde\psi = f \circ \tilde\phi$ for a continuous convex function $\tilde f : \mathbb{R} \to \mathbb{R}$ which is smooth except at one point (where $\zeta = \tilde\phi$) and satisfies $\tilde f(x) = x$ for $x \le a'$ with some $a' < a$. Let f be a smooth convex function which agrees with $\tilde f$ for $x \le a'$ and $x \ge a$. We can replace $\tilde\psi$ on $\{\tilde\phi \le a\} \setminus \tilde U$ by the smooth function $f \circ \tilde\phi$, without changing it near $D_{1+\varepsilon}$ and keeping it i-convex. Let us denote the resulting function by $\hat\psi$. Finally, we smoothen the function $\hat\psi$, without changing it on $\{\tilde\phi \le a\} \setminus \tilde U$ and near $D_{1+\varepsilon}$, to the desired i-convex function ψ.

It only remains to verify that the smoothing processes do not create new critical points. For the step from $\tilde\psi$ to $\hat\psi$ this is obvious. For the smoothing from $\hat\psi$ to ψ, after shrinking $\tilde U$ and $\tilde W$, we may assume that it takes place in the region where $0 < r < \gamma$ and $\tilde\phi(x,y) = \tilde\phi_{D_{1+\varepsilon}}(x_1,\dots,x_k) + Ar^2$. In this region we have $\nabla r \cdot \tilde\phi > 0$. Since ∇r is also transverse to the hypersurface $\Sigma = \{R^2 - Ar^2 = 1\}$, hence to each of the nearby hypersurfaces Σ_t, it satisfies $\nabla r \cdot \zeta > 0$ whenever $\zeta > b'$. Now it follows from Propositions 2.20 and 2.21 that the smoothing of $\max(\zeta, \tilde\phi)$ does not create new critical points. $\square$

7.6 Proof of the surrounding and extension theorems

Lemma 7.26. *Let (V,J) be a complex manifold of complex dimension n and $W \subset V$ a compact domain. Let $\Delta \subset V \setminus \mathrm{int} W$ be a real analytically embedded totally real k-ball attached J-orthogonally to ∂W along $\partial\Delta$. Let ν_Δ be a real analytic vector field along $\partial\Delta$, tangent to Δ and pointing out of Δ. Then there exists a holomorphic embedding $F : H_\varepsilon \hookrightarrow V$ with $F(D) = \Delta$ whose differential along S maps ν to ν_Δ and $T(\partial^- H)|_S$ to $T(\partial W)$.*

Proof. Pick a diffeomorphism $f_0 : D \to \Delta$ with $df_0 \cdot \nu = \nu_\Delta$ along ∂D. By Theorem 8.22, we can approximate f_0 by a real analytic diffeomorphism $f_1 : D \to \Delta$ still satisfying $df_1 \cdot \nu = \nu_\Delta$. Since Δ is attached J-orthogonally to ∂W, $g_1 := f_1|_{\partial D} : S \hookrightarrow \partial W$ is an isotropic embedding. Its conformally symplectic normal bundle $CSN(g_1) \subset g_1^*\xi$ extends as the complex normal bundle over D, hence it is trivial. By the remark after Proposition 3.18, f_1 extends to a totally real embedding $f_2 : D \times D_\varepsilon^{n-k} \hookrightarrow V \setminus \mathrm{int} W$ such that $g_2 := f_2|_{\partial D \times D_\varepsilon^{n-k}}$ is Legendrian. By Corollary 8.25, we can approximate g_2 by a real analytic Legendrian embedding g_3, keeping it fixed on $\partial D \times \{0\}$. Again using Theorem 8.22 (and possibly shrinking ε), we extend g_3 to a real analytic totally real embedding $f_3 : D \times D_\varepsilon^{n-k} \hookrightarrow V \setminus \mathrm{int} W$, still satisfying $df_3 \cdot \nu = \nu_\Delta$ along S. Now complexify f_3 to a holomorphic embedding $F : H_\varepsilon \hookrightarrow V$. By construction, $dF \cdot \nu = \nu_\Delta$. For the last condition, note that $T(\partial^- H)|_S = T(S \times D_\varepsilon^{n-k}) \otimes \mathbb{C} \oplus \mathbb{R} i\nu$. The differential dF at S maps $T(S \times D_\varepsilon^{n-k}) \otimes \mathbb{C}$ to $\xi \subset T(\partial W)$ because $F|_{S \times D_\varepsilon^{n-k}}$ is Legendrian, and $i\nu$ to $J\nu_\Delta \in T(\partial W)$ because Δ was attached J-orthogonally. $\square$

Now we are ready to prove the theorems of Section 7.1.

Proof of Theorem 7.2. By Lemma 7.26, there exists a holomorphic embedding $F : H_\varepsilon \hookrightarrow V$ with $F(D) = \Delta$ whose differential along S maps $T(\partial^- H)|_S$ to $T(\partial W)$ (the vector field ν_Δ is irrelevant here). The i-convex hypersurface $\Sigma := F^{-1}(\partial W) \subset H$ is tangent to the hypersurface $\{R^2 - Ar^2 = 1\}$ along $S = \{r = 0, R = 1\}$, for any $A > 1$. By Corollary 7.19 and Lemma 7.24, there exists an i-convex hypersurface $\Sigma' \subset F^{-1}(U)$ (given by $\{R = \varphi(r)\}$ for a suitable shape φ) which surrounds the disk D and coincides with Σ near ∂H. Let $\tilde{W} \subset H$ be the region bounded by Σ' and containing D. Then $W' := W \cup F(\tilde{W})$ is the desired neighbourhood.

The last assertion in Theorem 7.2 follows from Lemma 7.24. $\square$

Proof of Theorem 7.7. After a small perturbation near $\partial\Delta$, we may assume that ϕ is real analytic near $\partial\Delta$. Let ν_Δ be the unique vector field tangent to Δ along $\partial\Delta$ with $\nu_\Delta \cdot \phi = -2$. Thus ν_Δ is real analytic. By Lemma 7.26, there exists a holomorphic embedding $F : H_\varepsilon \hookrightarrow V$ with $F(D) = \Delta$ whose differential along S maps ν to ν_Δ and $T(\partial^- H)|_S$ to $T(\partial W)$. The i-convex function $F^*\phi : U \cup D \to \mathbb{R}$, with U a neighbourhood of S, satisfies $\phi|_S \equiv a$ and $d\phi = -2dR$ along S. Let $\tilde{\psi} : H \supset \tilde{W} \to \mathbb{R}$ be the i-convex function provided by Proposition 7.25. Property (c) allows us to extend $F_*\tilde{\psi}$ by $f \circ \phi$ to a J-convex function ψ on $W' := W \cup F(\tilde{W})$. The properties of ψ in Theorem 7.7 follow from the corresponding properties in Proposition 7.25. $\square$

Chapter 8

Real analytic approximations

8.1 Some complex analysis on Stein manifolds

There exist a number of equivalent definitions of a Stein manifold. We have already encountered two of them.

Affine definition. *A complex manifold V is Stein if it admits a proper holomorphic embedding into some $\mathbb{C}^N$.*

J-convex definition. *A complex manifold V is Stein if it admits an exhausting J-convex function $f : V \to \mathbb{R}$.*

The classical definition rests on the concept of holomorphic convexity. To a subset $K \subset V$ of a complex manifold associate its *holomorphically convex hull*

$$\hat{K} := \{x \in V \mid |f(x)| \leq \sup_K |f| \text{ for all holomorphic functions } f : V \to \mathbb{C}\}.$$

Call V *holomorphically convex* if $\hat{K}$ is compact for all compact subsets $K \subset V$.

Example 8.1. Let $B \subset \mathbb{C}^N$ be a closed ball around the origin. For $x \notin B$ the holomorphic function $f(z) := (z, x)$ satisfies $|f(z)| \leq |z|\,|x| < |x|^2 = |f(x)|$ for all $z \in B$. Hence $B = \hat{B}$ equals its own holomorphically convex hull.

Next consider a properly embedded complex submanifold $V \subset \mathbb{C}^{\mathbb{N}}$ and a compact subset $K \subset V$. Let $B \subset \mathbb{C}^N$ be a closed ball containing K. Then $\hat{K} \subset (V \cap B)\hat{} \subset \hat{B} = B$, where the first two holomorphically convex hulls are taken in V and the third in $\mathbb{C}^N$. Since $\hat{K}$ is closed in V, it is compact. This shows that V is holomorphically convex.

Example 8.2 (Hartogs phenomenon). The *Hartogs domain* $\Omega := \text{int}B^4(1) \setminus B^4(1/2) \subset \mathbb{C}^2$ has the holomorphically convex hull $\hat{\Omega} = \text{int}B^4(1)$ (in particular, Ω is not holomorphically convex). To see this, let $f : \Omega \to \mathbb{C}$ be a holomorphic function. For fixed $z \in \mathbb{C}$, $|z| < 1$, the function $w \mapsto f(z,w)$ on the annulus (or disk) $A_z := \{w \in \mathbb{C} \mid 1/4 - |z|^2 < |w|^2 < 1 - |z|^2\}$ has a Laurent expansion

$$f(z,w) = \sum_{k=-\infty}^{\infty} a_k(z) w^k.$$

The coefficients $a_k(z)$ are given by

$$a_k(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(z,\zeta)}{\zeta^{k+1}} d\zeta$$

for any $r > 0$ with $1/4 - |z|^2 < r^2 < 1 - |z|^2$. In particular, $a_k(z)$ depends holomorphically on z with $|z| < 1$. Since A_z is a disk for $|z| > 1/2$, we have $a_k(z) = 0$ for $k < 0$ and $|z| > 1/2$, hence by unique continuation for all z with $|z| < 1$. Thus the Laurent expansion defines a holomorphic extension of f to the ball $\mathrm{int} B^4(1)$.

Classical definition. *A complex manifold V is Stein if it has the following 3 properties:*

(i) V is holomorphically convex;

(ii) for any $x \neq y \in V$ there exists a holomorphic function $f : V \to \mathbb{C}$ with $f(x) \neq f(y)$;

(iii) for every $x \in V$ there exist holomorphic functions $f_1, \dots, f_n : V \to \mathbb{C}$ which form a holomorphic coordinate system at x.

Clearly, the affine definition implies the other two (holomorphic convexity was shown in Example 8.1). The classical definition immediately implies that every compact subset $K \subset V$ can be holomorphically embedded into some $\mathbb{C}^N$. The implication "classical $\Longrightarrow$ affine" is the content of

Theorem 8.3. *[Remmert [42]] A Stein manifold V in the classical sense admits a proper holomorphic embedding into some $\mathbb{C}^N$.*

Remark 8.4. A lot of research has gone into finding the smallest N for given $n = \dim_{\mathbb{C}} V$. After intermediate work of Forster, the optimal integer $N = [3n/2] + 1$ was finally established by Eliashberg-Gromov [11] and Schürmann [43].

The implication "J-convex $\Longrightarrow$ classical" was proved by Grauert in 1958:

Theorem 8.5 (Grauert [19])**.** *A complex manifold which admits an exhausting J-convex function is Stein in the classical sense.*

In particular, Grauert's theorem solves what was known, for domains in $\mathbb{C}^n$, as "Levi's problem":

Corollary 8.6. *A relatively compact domain $U \subset V$ in a Stein manifold V with smooth J-convex boundary ∂U is Stein.*

Proof. By Lemma 1.3, there exists a J-convex function $\phi : W \to (0,2)$ on a neighbourhood W of ∂U in V with $\partial U = \phi^{-1}(1)$. Let $\psi : V \to \mathbb{R}$ be a J-convex function with $\min_{\bar U}\psi > 0$. Pick a convex increasing diffeomorphism $f : (0,1) \to (0,\infty)$. Then a smoothing of $\max(f\circ\phi,\psi) : U \to \mathbb{R}$ is J-convex and exhausting, so U is Stein by Grauert's theorem. □

Remark 8.7. In fact, Grauert proves in [19] the following generalization of Levi's problem: A relatively compact domain $U \subset V$ in a complex (not necessarily Stein) manifold V with smooth J-convex boundary ∂U is holomorphically convex.

It is clear from any of the definitions that properly embedded complex submanifolds of Stein manifolds are Stein. We will refer to them as *Stein submanifolds*.

Two fundamental results about Stein manifolds are Cartan's Theorems A and B. They are formulated in the language of sheaves, see [7] for the relevant definitions and properties. Let V be a complex manifold and $\mathcal{O}$ the sheaf of holomorphic functions on V. For a nonnegative integer p, let $\mathcal{O}^p$ be the sheaf of holomorphic maps to $\mathbb{C}^p$. A sheaf $\mathcal{F}$ on V is called *analytic* if for each $x \in V$, $\mathcal{F}_x$ is a module over $\mathcal{O}_x$, and the multiplication $\mathcal{O}\times\mathcal{F}\to\mathcal{F}$ is continuous. A sheaf homomorphism $f : \mathcal{F} \to \mathcal{G}$ between analytic sheaves is called analytic if it is a module homomorphism. An analytic sheaf $\mathcal{F}$ is called *coherent* if every $x \in V$ has a neighbourhood U such that $\mathcal{F}_U$ equals the cokernel of an analytic sheaf homomorphism $f : \mathcal{O}^p_U \to \mathcal{O}^q_U$, for some nonnegative integers p, q.

Oka's Coherence Theorem [40] states that a subsheaf $\mathcal{F}$ of $\mathcal{O}^p$ is coherent if and only if it is *locally finitely generated*, i.e., for every point $x \in V$ there exists a neighbourhood U and finitely many sections f_i of $\mathcal{F}_U$ that generate $\mathcal{F}_y$ as an $\mathcal{O}_y$-module for every $y \in U$.

Example 8.8. Let $W \subset V$ be a properly embedded complex submanifold of a complex manifold V and $d \geq 0$ an integer. For an open subset $U \subset V$, let $\mathcal{I}_U$ be the ideal of holomorphic functions on U whose d-jet vanishes at all points of $U\cap W$. This defines an analytic sheaf $\mathcal{I}$ on V. We claim that $\mathcal{I}$ is coherent. To see this, let $x \in V$. If $x \notin W$ we find a neighbourhood U of x with $U \cap W = \emptyset$ (since $W \subset V$ is closed), hence $\mathcal{I}_U = \mathcal{O}_U$. If $x \in W$ we find a small open polydisk $U \cong \mathrm{int}\big(B^2(1)\times\cdots\times B^2(1)\big) \subset V$ around x with complex coordinates $(z_1,\dots,z_n)$ in which $W \cap U = \{z_1 = \cdots = z_k = 0\}$. Then the ideal I_U is generated as an $\mathcal{O}_U$-module by the monomials of degree $(d+1)$ in $z_1,\dots,z_k$, so by Oka's Coherence Theorem [40], $\mathcal{I}$ is coherent.

Remark 8.9. The coherence of the sheaf $\mathcal{I}$ in the preceding example can also be proved without Oka's theorem as follows. As above, let $(z_1,\dots,z_n)$ be complex coordinates on a polydisk U in which $W \cap U = \{z_1 = \cdots = z_k = 0\}$. We claim that every $f \in \mathcal{I}_U$ has a unique representation

$$f(z) = \sum_I f_I(z) z^I,$$

where the summation is over all $I = (i_1,\dots,i_k)$ with $i_1 + \cdots + i_k = d+1$ and

$z^I = z_1^{i_1} \dots z_k^{i_k}$. The coefficient f_I is a holomorphic function of $z_\ell, \dots, z_n$, where $1 \le \ell \le k$ is the largest integer with $i_\ell \neq 0$.

We first prove the claim for $d = 0$ by induction over k. The case $k = 1$ is clear, so let $k > 1$. The function $(z_k, \dots, z_n) \mapsto f(0, \dots, 0, z_k, \dots, z_n)$ vanishes at $z_k = 0$, thus (as in the case $k = 1$) it can be uniquely written as $z_k f_k(z_k, \dots, z_n)$ with a holomorphic function f_k. Since the function $(z_1, \dots, z_n) \mapsto f(z_1, \dots, z_n)$ vanishes at $z_1 = \dots = z_{k-1} = 0$, by induction hypothesis it can be uniquely written as $z_1 f_1(z_1, \dots, z_n) + \dots + z_{k-1} f_{k-1}(z_{k-1}, \dots, z_n)$ with holomorphic functions $f_1, \dots, f_{k-1}$. This proves the case $d = 0$. The general case $d > 0$ follows by induction over d: Using the case $d = 0$, we write $f(z)$ uniquely as $z_1 f_1(z_1, \dots, z_n) + \dots + z_k f_k(z_k, \dots, z_n)$. Now note that the functions $f_1, \dots, f_k$ must vanish to order $d-1$ at $z_1 = \dots = z_k = 0$ and use the induction hypothesis. This proves the claim.

By the claim, $\mathcal{I}_U$ is the direct sum of copies of the rings $\mathcal{F}_U^\ell$ of holomorphic functions of $z_\ell, \dots, z_n$ for $1 \le \ell \le k$. Since $\mathcal{F}_U^\ell$ is isomorphic to the cokernel of the homomorphism $\mathcal{O}_U^{\ell-1} \to \mathcal{O}_U$, $f_1, \dots, f_{\ell-1} \mapsto z_1 f_1 + \dots + z_{\ell-1} f_{\ell-1}$, this proves coherence of $\mathcal{I}$.

Now we can state Cartan's Theorems A and B. Denote by $H^q(V, \mathcal{F})$ the cohomology with coefficients in the sheaf $\mathcal{F}$. In particular, $H^0(V, \mathcal{F})$ is the space of sections in $\mathcal{F}$. Every subsheaf $\mathcal{G} \subset \mathcal{F}$ induces a long exact sequence

$$... \to H^q(V, \mathcal{G}) \to H^q(V, \mathcal{F}) \to H^q(V, \mathcal{F}/\mathcal{G}) \to H^{q+1}(V, \mathcal{G}) \to \dots .$$

Theorem 8.10 (Cartan [7])**.** *Let V be a Stein manifold and $\mathcal{F}$ a coherent analytic sheaf on V. Then*

(A) for every $x \in V$, $H^0(V, \mathcal{F})$ generates $\mathcal{F}_x$ as an $\mathcal{O}_x$-module;

(B) $H^q(V, \mathcal{F}) = \{0\}$ for all $q > 0$.

We will only use the following two consequences of Cartan's Theorem B.

Corollary 8.11. *Let W be a Stein submanifold of a Stein manifold V. Then every holomorphic function $f : W \to \mathbb{C}$ extends to a holomorphic function $F : V \to \mathbb{C}$. More generally, let $f : U \to \mathbb{C}$ be a holomorphic function on a neighbourhood of W and d a nonnegative integer. Then there exists a holomorphic function $F : V \to \mathbb{C}$ whose d-jet coincides with that of f at points of W.*

Proof. Let $\mathcal{I}$ be the analytic sheaf of holomorphic functions on V whose d-jet vanishes at points of W. By the example above, $\mathcal{I}$ is coherent. Thus by Cartan's Theorem B, $H^1(V, \mathcal{I}) = 0$, so by the long exact sequence the homomorphism $H^0(V, \mathcal{O}) \to H^0(V, \mathcal{O}/\mathcal{I})$ is surjective. Now $\mathcal{O}_x/\mathcal{I}_x = \{0\}$ for $x \notin W$, and for $x \in W$ elements of $\mathcal{O}_x/\mathcal{I}_x$ are d-jets of germs of holomorphic functions along W. So f defines a section in $\mathcal{O}/\mathcal{I}$, and we conclude that f is the restriction of a section F in $\mathcal{O}$. □

Corollary 8.12. *Every Stein submanifold W of a Stein manifold V is the common zero set of a finite number (at most $\dim_{\mathbb{C}} V+1$) of holomorphic functions $f_i : V \to \mathbb{C}$.*

Proof. The argument is given in [8]. It uses some basic properties of analytic subvarieties, see e.g. [21]. An *analytic subvariety* of a complex manifold V is a closed subset $Z \subset V$ that is locally the zero set of finitely many holomorphic functions. Z is a stratified space $Z = Z_0 \cup \cdots \cup Z_k$, where Z_i is a (non-closed) complex submanifold of dimension i. Define the (complex) dimension of Z as the dimension k of the top stratum. If $Z' \subset Z$ are analytic subvarieties of the same dimension, then Z' contains a connected component of the top stratum Z_k of Z.

Now let $W \subset V$ be a Stein submanifold of a Stein manifold V. Pick a set $S_1 \subset V$ containing one point on each connected component of $V \setminus W$. Since S_1 is discrete, $W \cup S_1$ is a Stein submanifold of V. By Corollary 8.11, there exists a holomorphic function $f_1 : V \to \mathbb{C}$ which equals 0 on W and 1 on S_1. The zero set $W_1 := \{f_1 = 0\}$ is an analytic subvariety of V, containing W, such that $W_1 \setminus W$ has dimension $\le n-1$, where $n = \dim_{\mathbb{C}} V$. Pick a set $S_2 \subset W_1 \setminus W$ containing one point on each connected component of the top stratum of W_1 that is not contained in W. Since each compact set meets only finitely many components of W_1, the set S_2 is discrete, so $W \cup S_2$ is a Stein submanifold of V. By Corollary 8.11, there exists a holomorphic function $f_2 : V \to \mathbb{C}$ which equals 0 on W and 1 on S_2. The zero set $W_2 := \{f_1 = f_2 = 0\}$ is an analytic subvariety of V, containing W, such that $W_2 \setminus W$ has dimension $\le n-2$. Continuing this way, we find holomorphic functions $f_1, \dots, f_{n+1} : V \to \mathbb{C}$ such that $W \subset W_{n+1} := \{f_1 = \cdots = f_{n+1} = 0\}$ and $W_{n+1} \setminus W$ has dimension ≤ -1. Thus $W_{n+1} \setminus W = \emptyset$ and $W = \{f_1 = \cdots = f_{n+1} = 0\}$. □

8.2 Real analytic approximations

In order to holomorphically attach handles, we need to approximate smooth objects by real analytic ones. In this section we collect the relevant results.

A function $f : U \to \mathbb{R}^m$ on an open domain $U \subset \mathbb{R}^n$ is called *real analytic* if it is locally near each point given by a convergent power series. A *real analytic manifold* is a manifold with an atlas such that all transition functions are real analytic. A submanifold is called real analytic if it is locally the transverse zero set of a real analytic function. Real analytic bundles and sections are defined in the obvious way.

Remark 8.13. As a special case of the Cauchy-Kowalewskaya theorem (see e.g. [14]), the solution of an ordinary differential equation with real analytic coefficients depends real analytically on all parameters.

Complexification. There is a natural functor, called *complexification*, from the real analytic to the holomorphic category. First note that any real analytic function $f : U \to \mathbb{R}^m$, defined on an open domain $U \subset \mathbb{R}^n$, can be uniquely extended to a holomorphic function $f^{\mathbb{C}} : U^{\mathbb{C}} \to \mathbb{C}^m$ on an open domain $U^{\mathbb{C}} \subset \mathbb{C}^n$ with $U^{\mathbb{C}} \cap \mathbb{R}^n = U$. A bit less obviously, any real analytic manifold M can be complexified to a complex manifold $M^{\mathbb{C}}$ which contains M as a real analytic submanifold. This can be seen as follows (see [5] for details). Pick a locally finite covering of M by countably many real analytic coordinate charts $\phi_i : \mathbb{R}^n \supset U_i \to M$. So the transition functions

$$\phi_{ij} := \phi_j^{-1} \circ \phi_i : U_{ij} := \phi_i^{-1}\Big(\phi_i(U_i) \cap \phi_j(U_j)\Big) \to U_{ji}$$

are real analytic diffeomorphisms. Successively extend them to biholomorphic maps $\phi_{ij}^{\mathbb{C}} : U_{ij}^{\mathbb{C}} \to U_{ji}^{\mathbb{C}}$ such that $\phi_{ji}^{\mathbb{C}} = (\phi_{ij}^{\mathbb{C}})^{-1}$. Note that $U_{ii}^{\mathbb{C}} = U_i^{\mathbb{C}}$ and $\phi_{ii}^{\mathbb{C}} = \mathbb{1}$. Define $M^{\mathbb{C}}$ as the quotient of the disjoint union $\coprod_i U_i^{\mathbb{C}}$ by the equivalence relation $z_i \sim z_j$ iff $z_i \in U_{ij}^{\mathbb{C}}$ and $z_j = \phi_{ij}^{\mathbb{C}}(z_i) \in U_{ji}^{\mathbb{C}}$. (This is an equivalence relation because of the cocycle condition $\phi_{jk}^{\mathbb{C}} \circ \phi_{ij}^{\mathbb{C}} = \phi_{ik}^{\mathbb{C}}$.) The inclusions $U_i^{\mathbb{C}} \hookrightarrow \coprod_j U_j^{\mathbb{C}}$ induce coordinate charts $U_i^{\mathbb{C}} \hookrightarrow M^{\mathbb{C}}$ with biholomorphic transition functions. Finally, this construction needs to be slightly modified to ensure that $M^{\mathbb{C}}$ is Hausdorff (see [5]).

Similarly, one sees that a real analytic map $f : M \to N$ between real analytic manifolds extends to a holomorphic map $f^{\mathbb{C}} : M^{\mathbb{C}} \to N^{\mathbb{C}}$ between (sufficiently small) complexifications. It follows that the complexification $M^{\mathbb{C}}$ is unique in the sense that if V, W are complex manifolds, containing M as real analytic and totally real submanifolds, with $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W = \dim_{\mathbb{R}} M$, then some neighbourhoods of M in V and W are biholomorphic. A corresponding uniqueness holds for complexifications of maps. As a real manifold, the complexification $M^{\mathbb{C}}$ is diffeomorphic to the tangent bundle TM.

Complexification has the obvious functorial properties. For example, if $N \subset M$ is a real analytic submanifold of a real analytic manifold M, then the (sufficiently small) complexification $N^{\mathbb{C}}$ is a complex submanifold of $M^{\mathbb{C}}$.

The crucial observation, due to Grauert [19], is that complexifications of real analytic manifolds are in fact Stein.

Proposition 8.14. *Let $M^{\mathbb{C}}$ be the complexification of a real analytic manifold M. Then M possesses arbitrarily small neighbourhoods in $M^{\mathbb{C}}$ which are Stein.*

Proof. By Proposition 1.9, M possesses arbitrary small neighbourhoods with exhausting J-convex functions. By Grauert's Theorem 8.5, these neighbourhoods are Stein. □

A complexification $M^{\mathbb{C}}$ which is Stein is called a *Grauert tube* of M. Now the basic results about real analytic manifolds follow via complexifiction from corresponding results about Stein manifolds.

Corollary 8.15. *Every real analytic manifold admits a proper real analytic embedding into some $\mathbb{R}^N$.*

Proof. By Theorem 8.3, a Grauert tube $M^{\mathbb{C}}$ of M embeds properly holomorphically into some $\mathbb{C}^N$. Then restrict this embedding to M. □

Corollary 8.16. *Let N be a properly embedded real analytic submanifold of a real analytic manifold M. Then every real analytic function $f : N \to \mathbb{R}$ extends to a real analytic function $F : M \to \mathbb{R}$. More generally, let $f : U \to \mathbb{R}$ be a real analytic function on a neighbourhood of N and d a nonnegative integer. Then there exists a real analytic function $F : M \to \mathbb{R}$ whose d-jet coincides with that of f at points of N.*

Proof. Let $M^{\mathbb{C}}$ be a Grauert tube of M. After possibly shrinking $M^{\mathbb{C}}$, we may assume that a complexification $N^{\mathbb{C}}$ of N is a properly embedded complex submanifold of $M^{\mathbb{C}}$, and f complexifies to a holomorphic function $f^{\mathbb{C}}$ on a neighbourhood of $N^{\mathbb{C}}$ in $M^{\mathbb{C}}$. Corollary 8.11 provides a holomorphic function $G : M^{\mathbb{C}} \to \mathbb{C}$ whose d-jet agrees with that of $f^{\mathbb{C}}$ at points of $N^{\mathbb{C}}$. Then the restriction of the real part of G to M is the desired function F. □

Corollary 8.17. *Every properly embedded real analytic submanifold N of a real analytic manifold M is the common zero set of a finite number (at most $2\dim_{\mathbb{R}} M + 2$) of real analytic functions $f_i : M \to \mathbb{R}$.*

Proof. Complexify N to a properly embedded submanifold $N^{\mathbb{C}} \subset M^{\mathbb{C}}$ of a Grauert tube $M^{\mathbb{C}}$. By Corollary 8.12, $N^{\mathbb{C}}$ is the zero set of at most $n + 1$ holomorphic functions $F_i : M^{\mathbb{C}} \to \mathbb{C}$, where $n = \dim_{\mathbb{R}} M$. The restrictions of $\operatorname{Re} F_i$ and $\operatorname{Im} F_i$ to M yield the desired functions f_i. □

Remark 8.18. H. Cartan [8] takes a slightly different route to prove Corollaries 8.16 and 8.17: Define coherent analytic sheaves on real analytic manifolds analogously to the complex analytic case. Cartan proves that for every coherent analytic sheaf $\mathcal{F}$ on M, there exists a coherent analytic sheaf $\mathcal{F}^{\mathbb{C}}$ on a complexification $M^{\mathbb{C}}$ such that $\mathcal{F}^{\mathbb{C}}|_M = \mathcal{F} \otimes \mathbb{C}$. From this he deduces the analogues of Theorems A and B in the real analytic category, which imply the corollaries as in the complex analytic case.

Corollary 8.15 implies that every C^k-function on a real analytic manifold M can be C^k-approximated by real analytic functions. To state the result, equip M with a metric and connection so that we can speak of k-th (covariant) derivatives of functions on M and their norms.

Corollary 8.19. *Let $f : M \to \mathbb{R}$ be a C^k-function on a real analytic manifold. Then for every compact subset $K \subset M$ and $\varepsilon > 0$ there exists a real analytic function $g : M \to \mathbb{R}$ which is ε-close to f together with its first k derivaties on K.*

Proof. Embed M real analytically into some $\mathbb{R}^N$. Pick any C^k-function $F : \mathbb{R}^N \to \mathbb{R}$ which coincides with f on K. By Weierstrass' theorem (see e.g. [16]), F can be C^k-approximated over K by a polynomial $G : \mathbb{R}^N \to \mathbb{R}$. Let g be the restriction of G to M. □

On the other hand, Corollary 8.16 shows that every real analytic function on a properly embedded real analytic submanifold N of a real analytic manifold M can be extended to a real analytic function on M, with prescribed normal d-jet along N. The following result combines the approximation and extension results.

Proposition 8.20. *Let $f : M \to \mathbb{R}$ be a C^k-function on a real analytic manifold. Let N be a properly embedded real analytic submanifold, $K \subset M$ a compact subset, d a nonnegative integer and $\varepsilon > 0$. Suppose that f is real analytic on a neighbourhood of N. Then there exists a real analytic function $F : M \to \mathbb{R}$ with the following properties:*

- *F is ε-close to f together with its first k derivatives over K;*
- *the d-jet of F coincides with that of f at every point of N.*

The proof is based on the following

Lemma 8.21. *For every $d, k \in \mathbb{N}$ there exists a constant $C_{d,k}$ such that for all $p \in \mathbb{N}$, $D > \delta > 0$ and $\gamma > 0$ there exists a polynomial $P : \mathbb{R} \to \mathbb{R}$ with the following properties:*

- *$P(0) = 1$ and $P'(0) = \cdots = P^{(d)}(0) = 0$;*
- *$|P^{(l)}(x)| \leq \gamma$ for all $0 \leq l \leq k$ and $\delta \leq |x| \leq D$;*
- *$|P^{(l)}(x)| \leq C_{d,k}/\delta^l$ for all $0 \leq l \leq k$ and $|x| \leq \delta$.*

Proof. Let k be given. Pick a C^k-function $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $f(x) = 0$ near $x = 0$;
- $f(x) = 1$ for $|x| \geq 1$;
- $|(f-1)^{(l)}(x)| \leq C_k/2$ for $|x| \leq 1$ and $0 \leq l \leq k$,

with a constant C_k depending only on k. For $D > \delta > 0$ define $g(x) := f(x/\delta)$. Then $g : \mathbb{R} \to \mathbb{R}$ has the following properties:

- $g(x) = 0$ near $x = 0$;
- $g(x) = 1$ for $\delta \leq |x| \leq D$;
- $|(g-1)^{(l)}(x)| \leq C_k/(2\delta^l)$ for $|x| \leq \delta$ and $0 \leq l \leq k$.

By Weierstrass' theorem (see e.g. [16]), we find for every $\beta > 0$ a polynomial $Q : \mathbb{R} \to \mathbb{R}$ satisfying

- $Q(0) = 0$;

- $|(Q-1)^{(l)}(x)| \le \beta$ for $\delta \le |x| \le D$ and $0 \le l \le k$;
- $|(Q-1)^{(l)}(x)| \le C_k/\delta^l$ for $|x| \le \delta$ and $0 \le l \le k$.

For $d \in \mathbb{N}$ consider the polynomial $P(x) := Q(x)^{d+1} - 1$. By the Leibniz rule,

$$(Q^{d+1})^{(l)}(x) = \sum_{i_1+\cdots+i_{d+1}=l} \binom{l}{i_1 \dots i_{d+1}} Q^{(i_1)}(x) \cdots Q^{(i_{d+1})}(x).$$

This shows $P(0) = 1$ and $P^{(l)}(0) = 0$ for $1 \le l \le d$. Now let $\gamma > 0$ be given. For $\delta \le |x| \le D$ and $1 \le l \le k$ the estimates on Q yield

$$\begin{aligned} |(P)^{(l)}(x)| &\le \sum_{i_1+\cdots+i_{d+1}=l} \binom{l}{i_1 \dots i_{d+1}} \beta^l (1+\beta)^{d+1} \\ &= (d+1)^l \beta^l (1+\beta)^{d+1} \le (d+1)^k \beta (1+\beta)^{d+1} \le \gamma \end{aligned}$$

for β sufficiently small. For $\delta \le |x| \le D$ and $l = 0$ we find

$$|P^{(l)}(x)| = |Q(x) - 1|\; |1 + Q(x) + \cdots + Q^d(x)| \le \beta(2d+1) \le \gamma$$

for $\beta \le 1$ sufficiently small. Similarly, for $|x| \le \delta$ and $1 \le l \le k$ we get

$$|(P)^{(l)}(x)| \le \sum_{i_1+\cdots+i_{d+1}=l} \binom{l}{i_1 \dots i_{d+1}} \frac{(C_k+1)^{d+1}}{\delta^l} \le \frac{(d+1)^k (C_k+1)^{d+1}}{\delta^l},$$

and for $|x| \le \delta$ and $l = 0$,

$$|P(x)| \le |Q^{d+1}(x)| + 1 \le (C_k+1)^{d+1} + 1.$$

Hence P satisfies the required estimates with $C_{d,k} := (d+1)^k (C_k+1)^{d+1} + 1$. □

Proof of Proposition 8.20. By Corollary 8.17, there exist real analytic functions $\phi_1, \dots, \phi_m : M \to \mathbb{R}$ such that $N = \{\phi_1 = \cdots = \phi_m = 0\}$. Then $\phi := \phi_1^2 + \cdots + \phi_m^2 : M \to \mathbb{R}$ is real analytic and $N = \phi^{-1}(0)$. Let dist_N be the distance from N with respect to some Riemannian metric on M. Since ϕ vanishes only to finite order in directions transversal to N, there exists an $r \in \mathbb{N}$ such that, after rescaling the metric, we have $\phi(x) \ge \mathrm{dist}_N(x)^r$ for all $x \in K$. Set $D := \max_K \phi$. For δ sufficiently small, $W := \{\phi \le \delta\}$ is a tubular neighbourhood of N over K.

For $\delta, \gamma > 0$ let $P : \mathbb{R} \to \mathbb{R}$ be the polynomial from Lemma 8.21. The real analytic function $\psi := P \circ \phi : M \to \mathbb{R}$ has the following properties:

- $\psi(x) =$ and $\psi'(x) = \cdots = \psi^{(d)}(x) = 0$ for $x \in N$;
- $|\psi^{(l)}(x)| \le C_1 \gamma$ for all $0 \le l \le k$ and $x \in K \setminus W$;
- $|\psi^{(l)}(x)| \le C_1 C_{d,k}/\delta^l$ for all $0 \le l \le k$ and $x \in W \cap K$,

with a constant C_1 depending only on ϕ (and not on δ and γ). Here and in the following we assume $M \subset \mathbb{R}^N$ and denote by $\psi^{(l)}$ any partial derivative of order l.

Without loss of generality, we may increase d until $d \geq r(k+1)$. By Corollary 8.16, there exists a real analytic function $h : M \to \mathbb{R}$ whose d-jet agrees with that of f at points of N. Hence there exists a constant C_2, depending only on f and h, such that

- $|(f-h)^{(l)}(x)| \leq C_2 \operatorname{dist}_N(x)^{d-l} \leq C_2\, \delta^{(d-l)/r}$ for all $0 \leq l \leq k$ and $x \in W \cap K$;
- $|(f-h)^{(l)}(x)| \leq C_2$ for all $0 \leq l \leq k$ and $x \in K \setminus W$.

We can estimate the product $\psi \cdot (f-h)$ for $0 \leq l \leq k$ and $x \in W \cap K$ by

$$\begin{aligned} |[\psi(f-h)]^{(l)}(x)| &\leq \sum_{i=0}^{l} |(f-h)^{(i)}(x)|\, |\psi^{(l-i)}(x)| \\ &\leq \sum_{i=0}^{l} C_2\, \delta^{(d-i)/r} \frac{C_1 C_{d,k}}{\delta^{l-i}} \leq (k+1) C_1 C_2 C_{d,k} \delta, \end{aligned}$$

since the exponent of δ satisfies $(d-i)/r + i - l \geq d/r - l \geq 1$ by the choice of d. Similarly, for $0 \leq l \leq k$ and $x \in K \setminus W$ we obtain

$$|[\psi(f-h)]^{(l)}(x)| \leq (k+1) C_1 C_2 \gamma.$$

Now let $\varepsilon > 0$ be given. By Corollary 8.19, there exists a real analytic function $g : M \to \mathbb{R}$ with $|(f-g)^{(l)}(x)| < \varepsilon'$ for all $0 \leq l \leq k$ and $x \in K$, with $\varepsilon' > 0$ to be determined later. Define the real analytic function

$$F := g + \psi \cdot (h-g) : M \to \mathbb{R}.$$

Since $\psi'(x) = \cdots = \psi^{(d)}(x) = 0$ for $x \in N$, the d-jet of F agrees with that of f at points of N. For $0 \leq l \leq k$ and $x \in K \setminus W$ we estimate

$$\begin{aligned} |(F-f)^{(l)}(x)| &\leq |[(1-\psi)(g-f)]^{(l)}(x)| + |[\psi(h-f)]^{(l)}(x)| \\ &\leq (1 + C_1\gamma)\varepsilon' + (k+1) C_1 C_2 \gamma. \end{aligned} \tag{8.1}$$

For $0 \leq l \leq k$ and $x \in W \cap K$ we find

$$\begin{aligned} |(F-f)^{(l)}(x)| &\leq |[(1-\psi)(g-f)]^{(l)}(x)| + |[\psi(h-f)]^{(l)}(x)| \\ &\leq \frac{C_1 C_{d,k}}{\delta^k} \varepsilon' + (k+1) C_1 C_2 C_{d,k} \delta. \end{aligned} \tag{8.2}$$

Now first choose $\gamma > 0$ small enough so that the second term on the right-hand side of (8.1) becomes $< \varepsilon/2$. Given γ, choose $\delta > 0$ small enough so that the second term on the right-hand side of (8.2) becomes $< \varepsilon/2$. Finally, choose $\varepsilon' > 0$ small enough so that the first terms on the right-hand sides of (8.1) and (8.2) become $< \varepsilon/2$. Then F has the desired properties. □

Proposition 8.20 clearly generalizes to sections in real analytic bundles $E \to M$. For this, view the total space of the bundle as a real analytic manifold and note that a map $M \to E$ that is C^0-close to a section is a section. Thus we have

Theorem 8.22. *Let $f : M \to E$ be a C^k-section in a real analytic fibre bundle $E \to M$ over a real analytic manifold M. Let $N \subset M$ be a properly embedded real analytic submanifold, $K \subset M$ a compact subset, d a nonnegative integer and $\varepsilon > 0$. Suppose that f is real analytic on a neighbourhood of N. Then there exists a real analytic section $F : M \to E$ with the following properties:*

(i) F is ε-close to f together with its first k derivatives over K;

(ii) the d-jet of F coincides with that of f at every point of N.

Example 8.23. Every Riemannian metric on a real analytic manifold can be C^k-approximated by a real analytic metric. By Remark 8.13, the exponential map of a real analytic metric is real analytic. Now the standard proof yields real analytic tubular resp. collar neighbourhoods of compact real analytic submanifolds resp. boundaries. In particular, this allows us to extend any compact real analytic manifold with boundary to a slightly larger open real analytic manifold.

Theorem 8.22 also has a version with parameters.

Corollary 8.24. *Let $E \to M$ be a real analytic fibre bundle over a real analytic manifold M, $K \subset M$ a compact subset, and $\varepsilon > 0$. Let $f_t : M \to E$ be a family of C^k-sections depending in a C^k fashion on a parameter t in a compact real analytic manifold T with boundary. Suppose that the f_t are real analytic for $t \in \partial T$ and depend real analytically on $t \in \partial T$. Then there exists a family of real analytic sections $F_t : M \to E$, depending real analytically on $t \in T$, with the following properties:*

(i) F_t is ε-close to f_t together with its first k derivatives over K for all $t \in T$;

(ii) $F_t = f_t$ for $t \in \partial T$.

Proof. By Example 8.23, we can include Λ in a larger open real analytic manifold $\tilde{\Lambda}$. Extend f_t to a C^k-family $\tilde{f}_t$ over $\tilde{\Lambda}$ and view $\tilde{f}_t$ as a C^k-section in the bundle $E \to \tilde{\Lambda} \times M$. Now apply Theorem 8.22 to this section, the compact set $\Lambda \times K$, and the properly embedded real analytic submanifold $\partial\Lambda \times M$. □

We conclude this chapter with a result on real analytic approximations of isotropic submanifolds in contact manifolds that will be needed later. See Chapter **??** for the relevant definitions.

Corollary 8.25. *Let Λ be a closed isotropic C^k-submanifold ($k \geq 1$) in a real analytic closed contact manifold (M, α) (i.e., the manifold M and the 1-form α are both real analytic). Then there exists a real analytic isotropic submanifold $\Lambda' \subset (M, \alpha)$ arbitrarily C^k-close to Λ.*

Similarly, let $(\Lambda_t)_{t\in[0,1]}$ *be a* C^k*-isotopy of closed isotropic* C^k*-submanifolds in* (M, α) *such that* Λ_0 *and* Λ_1 *are real analytic. Then there exists a real analytic isotopy of real analytic isotropic submanifolds* Λ'_t*, arbitrarily* C^k*-close to* Λ_t*, with* $\Lambda'_0 = \Lambda_0$ *and* $\Lambda'_1 = \Lambda_1$*.*

Proof. Let $\tilde\Lambda \subset M$ be a real analytic submanifold C^k-close to Λ, but not necessarily isotropic. Then $\Lambda = \phi(\tilde\Lambda)$ for a C^k-diffeomorphism $\phi : M \to M$ that is C^k-close to the identity. The contact form $\phi^*\alpha$ vanishes on $\tilde\Lambda$ but need not be real analytic. Thus $\phi^*\alpha$ induces a C^k-section in the real analytic vector bundle $T^*M|_{\tilde\Lambda} \to \tilde\Lambda$ which vanishes on the real analytic subbundle $T\tilde\Lambda \subset T^*M|_{\tilde\Lambda}$. Let $\nu \to \tilde\Lambda$ be the normal bundle to $T\tilde\Lambda$ in $T^*M|_{\tilde\Lambda}$ with respect to a real analytic metric and denote by $(\phi^*\alpha)^\nu$ the induced C^k-section in ν. Let β^ν be a real analytic section of ν that is C^k-close to $(\phi^*\alpha)^\nu$ and extend it to a real analytic section β of $T^*M|_{\tilde\Lambda}$ that vanishes on $T\tilde\Lambda$, and hence is C^k-close to $\phi^*\alpha$ along $\tilde\Lambda$. Extend β to a C^k one-form on M (still denoted by β) that is C^k-close to $\phi^*\alpha$. By construction, β is real analytic along $\tilde\Lambda$ and $\beta|_{\tilde\Lambda} = 0$.

By Theorem 8.22 (with $d = 0$), there exists a real analytic 1-form $\tilde\alpha$ that is C^k-close to β and coincides with β along $\tilde\Lambda$. In particular, $\tilde\alpha|_{\tilde\Lambda} = 0$. By construction, $\tilde\alpha$ is C^k-close to α. Hence $\alpha_t := (1-t)\tilde\alpha + t\alpha$ is a real analytic homotopy of real analytic contact forms. By Gray's stability theorem (Theorem 3.21), there exists an isotopy of diffeomorphisms $\phi_t : M \to M$ and positive functions f_t with $\phi_t^*\alpha = f_t\tilde\alpha$. Now in Moser's proof of Gray's stability theorem (see e.g. [6]), the ϕ_t are constructed as solutions of an ODE whose coefficients are real analytic and C^k-small in this case. Hence by Remark 8.13 the ϕ_t are real analytic, C^k-close to the identity, and depend real analytically on t. It follows that $\Lambda' := \phi_1(\tilde\Lambda)$ is real analytic, C^k-close to Λ, and $\alpha|_{\Lambda'} = 0$. □

Remark 8.26. (1) Corollary 8.25 remains valid (with essentially the same proof) if the submanifold Λ is not closed, providing a real analytic approximation on a compact subset $K \subset \Lambda$.

(2) If Λ is Legendrian, then Λ' is Legendrian isotopic to Λ: By the Legendrian neighbourhood theorem (Proposition 3.18), Λ' is the graph of the 1-jet of a function f in $J^1\Lambda$, and the functions tf provide the isotopy.

Chapter 9

Extension of Stein structures over handles

9.1 Handles in the holomorphic category

For the purposes of this section, let us slightly modify the definition of an attaching map. Let W be a manifold with boundary and extend it to a slightly larger manifold $\tilde{W}$. An *attaching map* is an embedding $F : H \supset U \hookrightarrow \tilde{W}$ such that $F(S) \subset \partial W$ and the differential dF along S maps $\partial^- H|_S$ to ∂W and the outward pointing vector field η to an inward pointing vector field η_F. Then for $\varepsilon > 0$ small let

$$W \cup_F H := W \amalg H/H \cap F^{-1}(W) \ni x \sim F(x) \in W \cap F(H)$$

Note that $F^{-1}(\partial W)$ is a graph over $\partial^- H$ near S, so $W \cup_F H$ describes indeed the attaching of a handle for ε small.

Remark 9.1. The following facts are seen as in the previous section.

(1) If J, J_H are almost complex structures on W, H and dF is complex linear along S, then $W \cup_F H$ carries a natural homotopy class of almost complex structures $J \cup_F H$ that agree with J on W and with J_H along D.

(2) An isotopy of attaching maps F_t, induces a canonical family of diffeomorphisms $\phi_t : W \cup_{F_0} H \to W \cup_{F_t} H$. Moreover, if the differentials dF_t are complex linear along S for almost complex structures J, J_t on W, H, then $\phi_t^*(J \cup_{F_t} J_t)$ is a continuous homotopy of almost complex structures on $W \cup_{F_0} H$.

(3) If J is an (integrable) complex structure on W and the attaching map $F : (U, i) \to (\tilde{W}, J)$ is holomorphic, then $W \cup_F H$ carries a natural (integrable) complex structure. We will consider below the case that ∂W is J-convex. Since $\partial^- H$ is Levi-flat for the complex structure i, the attaching map cannot map $\partial^- H$ to ∂W in that case. This explains our modified definition of "attaching map".

Lemma 9.2. *Let (W, J) be a complex manifold with real analytic J-convex boundary. Let $F_0 : H_\varepsilon \supset U_\varepsilon \hookrightarrow \tilde{W}$ be an attaching map such that $dF_0 : (TH|_S, i) \to (TW, J)$ is complex linear. Then (after shrinking ε) there exists a family of attaching maps $F_t : U_\varepsilon \hookrightarrow \tilde{W}$, $t \in [0,1]$, C^∞-close to F_0, such that F_1 is holomorphic and*

$$dF_t : (TH|_S, i) \to (TW, J)$$

is complex linear for all $t \in [0,1]$.

Proof. Without further mention, we will shrink ε whenever necessary. Moreover, all homotopies will be chosen C^∞-close to the original data.

As J integrable and ∂W is real analytic and J-convex, the maximal tangency ξ on ∂W is a real analytic contact structure. Set $P_\varepsilon := \partial D_1^k \times D_\varepsilon^{n-k}$ and consider the Legendrian embdding $g_0 := (F_0)|_{P_\varepsilon} : P_\varepsilon \hookrightarrow \partial W$. By Corollary 8.24 and the remark following it, there exists a Legendrian isotopy $g_t : P_\varepsilon \hookrightarrow \partial W$ such that g_1 is real analytic.

By hypothesis, dF_0 maps the vector field v along S to a vector field v_0 on ∂W transverse to ξ. By Theorem 8.22, there exists a family v_t of transverse vector fields on ∂W such that v_1 is real analytic. Set $\eta_t := Jv_t$. Again by Theorem 8.22, we can extend g_1 to a real analytic embedding $f_1 : (V_\varepsilon := D_{1+\varepsilon}^k \setminus \mathrm{int} D_1^k) \times D_\varepsilon^{n-k} \to \tilde{W}$ with $df_1 \cdot \eta = \eta_1$. Connect $f_0 := (F_0)|_{V_\varepsilon}$ to f_1 by a smooth isotopy of totally real embeddings $f_t : V_\varepsilon \to \tilde{W}$ with $df_t \cdot \eta = \eta_t$.

Complexify the (totally real) differentials df_t along S to complex linear isomorphisms $d^{\mathbb{C}} f_t : (TH|_S, i) \to (TW|_{f_t(S)}, J)$. Complexify the totally real embedding $f_1 : V_\varepsilon = U_\varepsilon \cap \mathbb{R}^n \hookrightarrow \tilde{W}$ to a holomorphic embedding $F_1 : U_\varepsilon \hookrightarrow \tilde{W}$. Note that $dF_0 = d^{\mathbb{C}} f_0$ and $dF_1 = d^{\mathbb{C}} f_1$ along S. Connect F_0 to F_1 by an isotopy of smooth embeddings $F_t : U_\varepsilon \hookrightarrow \tilde{W}$ with $dF_t = d^{\mathbb{C}} f_t$ along S. By construction, $dF_t = d^{\mathbb{C}} f_t$ maps $T(\partial^- H)|_S$ to $T(\partial W)$ and η to the inward pointing vector field η_t. Thus the F_t are attaching maps with $dF_t : (TH|_S, i) \to (TW, J)$ complex linear for all $t \in [0,1]$. □

Proposition 9.3. *Let (W, J) be a compact almost complex manifold of complex dimension $n > 2$ with boundary $\partial W = \partial^- W \cup \partial^+ W$ (we allow $\partial^- W = \emptyset$). Suppose W carries a function which is constant on the boundary components and has a unique critical point of index $k \le n$. Suppose that near $\partial^- W$, J is integrable and $\partial^- W$ is J-convex.*

Then there exists an integrable complex structure $\tilde{J}$ on W such that $\tilde{J} = J$ near $\partial^- W$ and $\tilde{J} \sim J$ rel $\partial^- W$.

Proof. Let $W' \subset W$ be a collar neighbourhood of $\partial^- W = \partial^- W'$ with real analytic J-conxev boundary $\partial^+ W'$. By Morse theory [37], there exists an embedding $f : H \hookrightarrow W$ of a k-handle, with attaching map $f_0 := f|_U : U \hookrightarrow W'$, such that W smoothly deformation retracts onto a smoothing of $W' \cup f(H)$. Let $J_0 := f^* J$ on H. By Proposition **??** and Lemma 9.2, there exists a family of almost complex structures J_t on H and an isotopy of attaching maps $f_t : U \hookrightarrow \tilde{W}'$

such that $J_1 = i$, f_1 is holomorphic, and $df_t : (TH|_S, J_t) \to (TW|_{f_t(S)}, J)$ is a complex isomorphism for all t. By Lemma **??**, this gives rise to a homotopy of almost complex structures J_t' on $W' \cup f(H)$, fixed near $\partial^- W$, such that $J_0' = J$ and $\tilde{J}_1 =: J'$ is integrable.

It only remains to extend J' to all of W. For this, let $\tilde{W} \subset W' \cup f(H)$ be a tubular neighbourhood of $W' \cup f(D)$. Let $g_t : \tilde{W} \hookrightarrow W$ be an isotopy of embeddings such $g_t = 1\!\!1$ near $W' \cup f(D)$, g_0 is the inclusion, and g_1 is a diffeomorphism. Now $\tilde{J} := g_{1*}J'$ is an integrable complex structure on W which coincides with J' on W'. Moreover, $\tilde{J}_t := g_{1*}g_t^* J'$ provides a homotopy rel W' from $\tilde{J}_0 = \tilde{J}$ to $\tilde{J}_1 = J'$. Since J' was homotopic rel $\partial^- W$ to J, this concludes the proof. □

9.2 Extension of Stein structures over handles

Theorem 9.4. *Let (W, J) be a compact almost complex manifold of complex dimension $n > 2$ with boundary $\partial W = \partial^- W \cup \partial^+ W$ (we allow $\partial^- W = \emptyset$). Let $\phi : W \to [a, b]$ be a function with $\partial^- W = \phi^{-1}(a)$, $\partial^+ W = \phi^{-1}(b)$ and a unique critical point in W of index $k \leq n$. Suppose that near $\partial^- W$, J is integrable and ϕ is J-convex.*

Then there exists an integrable complex structure $\tilde{J}$ on W such that $\tilde{J} = J$ near $\partial^- W$, $\tilde{J} \sim J$ rel $\partial^- W$, and ϕ is $\tilde{J}$-convex.

Proof. Let $\partial^- W \times [0, 1]$ be a collar neighbourhood of $\partial^- W = \partial^- W \times \{0\}$ on which J is integrable and ϕ is J-convex with level sets $\partial^- W \times \{t\}$. Let ϕ' be C^2-close to ϕ, real analytic near $\partial^- W \times \{1/2\}$, with $\phi' = \phi$ outside $\partial^- W \times [1/4, 3/4]$. Then ϕ' is J-convex and $\phi' = f^*\phi$ for a diffeomorphism f isotopic to the identity rel $W \setminus \partial^- W \times [1/4, 3/4]$. Thus it suffices to prove the theorem for ϕ' and $J' := f^* J$. Denoting ϕ', J' again by ϕ, J, we may hence assume that ϕ is real analytic near a level set $\phi^{-1}(a')$, $a' > a$, and J-convex on $W' := \phi^{-1}([a, a'])$.

By Proposition 9.3, J is homotopic rel $\partial^- W$ to an integrable complex structure J'. Perturb the gradient vector field $\nabla_{g_\phi}\phi$, fixed near $\partial^+ W$, to a C^1-close vector field X. Then X is gradient-like for ϕ and has a nondegenerate zero at the critical point p of ϕ. Let $\Delta \subset W \setminus \text{int} W'$ be the stable disk of p for X. Then Δ is totally real and real analytic. Moreover, since $X = \nabla_{g_\phi}\phi$ near $\partial^+ W'$, Δ is attached J'-orthogonally to $\partial^+ W'$ along $\partial\Delta$.

By Theorem 7.7, there exists a surjective J'-convex function $\psi : \tilde{W} \to [a, b]$ on a neighbourhood $\tilde{W}$ of $W' \cup \Delta$ with $\psi = \phi$ on W' and a unique index k critical point at p. Moreover, there exists an isotopy $h_t : \Delta \to \Delta$, fixed near $\partial\Delta$ and p, with $h_0 = 1\!\!1$ and $h_1^*\phi = \psi$. Now we argue as in the proof of Proposition 9.3. Extend h_t to an isotopy of embeddings $\tilde{h}_t : \tilde{W} \hookrightarrow W$ such that $\tilde{h}_t|_{W'} = 1\!\!1$, $\tilde{h}_t|_\Delta = h_t$, $\tilde{h}_0$ is the inclusion, and $\tilde{h}_1$ is a diffeomorphism. Then the Morse functions $\tilde{h}_1^*\phi, \psi : \tilde{W} \to [a, b]$ coincide on $W' \cup \Delta$. By Lemma 6.8, there exists an isotopy of diffeomorphisms $g_t : \tilde{W} \to \tilde{W}$, fixed on $W' \cup \Delta$, with $g_0 = 1\!\!1$ and

$g_1^*\tilde{h}_1^*\phi = \psi$. Hence the embeddings $f_t := \tilde{h}_t \circ g_t$ satisfy: f_0 is the inclusion, $f_t|_{W'} = \mathbb{1}$, and f_1 is a diffeomorphism with $f_{1*}\psi = \phi$. Now $\tilde{J} := f_{1*}J'$ is an integrable complex structure on W, which coincides with J' on W', such that ϕ is $\tilde{J}$-convex. Moreover, $J_t := f_{1*}f_t^*J'$ provides a homotopy rel W' from $J_0 = \tilde{J}$ to $J_1 = J'$. Since J' was homotopic rel $\partial^- W$ to J, this concludes the proof of Theorem 9.4 □

Now we are ready to prove the existence theorem for Stein structures stated in the introduction.

Theorem 9.5 (Eliashberg [9])**.** *Let V^{2n} be an open smooth manifold of dimension $2n > 4$ with an almost complex structure J and an exhausting Morse function ϕ without critical points of index $> n$. Then V admits a Stein structure. More precisely, J is homotopic through almost complex structures to an integrable complex structure $\tilde{J}$ such that ϕ is $\tilde{J}$-convex.*

Proof. Let $c_1 < c_2 < \dots$ be the critical levels of ϕ (possibly infinitely many). For simplicity, suppose that each critical level c_k carries a single critical point p_k; the obvious modifications for several critical points on one level are left to the reader. Let d_k be regular levels with

$$c_1 < d_1 < c_2 < d_2 < \dots$$

and set $V_k := \{\phi \le d_k\}$. We will inductively construct almost complex structures J_k, $k \in \mathbb{N}$, and homotopies J_k^t, $t \in [0,1]$, on V with the following properties:

- $J_k|_{V_k}$ is integrable and $\phi|_{V_k}$ is J_k-convex;
- $J_k^0 = J_{k-1}$, $J_k^1 = J_k$, and $J_k^t|_{V_{k-1}} = J_{k-1}$ for all $t \in [0,1]$.

Here we have set $J_0 := J$ and $V_0 := \emptyset$. The case $k = 1$ follows directly from Theorem 9.4 with $\partial^- W = \emptyset$. For the induction step, supppose that J_{k-1} and J_{k-1}^t have already been constructed. After replacing d_{k-1} by a slightly higher level in the preceding step, we may assume that J_{k-1} is integrable on a neighbourhood of V_{k-1}. Applying Theorem 9.4 to $W := V_k \setminus \mathrm{int}V_{k-1}$ and the almost complex structure J_{k-1}, we find a homotopy of almost complex structures $\tilde{J}_k^t$ on V_k such that $\tilde{J}_k^t|_{V_{k-1}} = J_{k-1}$ for all t, $\tilde{J}_k^0 = J_{k-1}$, $\tilde{J}_k = \tilde{J}_k^1$ is integrable, and $\phi|_{V_k}$ is $\tilde{J}_k$-convex. Let $\partial V_k \times [0,1]$ be a collar neighbourhood of $\partial V_k \cong \partial V_k \times \{0\}$ in $V \setminus \mathrm{int}V_k$ and extend $\tilde{J}_k^t$ to V by

$$J_k^t := \begin{cases} \tilde{J}_k^t & \text{on } V_k, \\ \tilde{J}_k^{t(1-s)} & \text{on } \partial V_k \times \{s\}, \\ J_{k-1} & \text{on } V \setminus (V_k \cup \partial V_k \times [0,1]). \end{cases}$$

This proves the induction step.

Now let sequences J_k, J_k^t as above be given. Since $J_k|_{V_{k-1}} = J_{k-1}$, the J_k fit together to an integrable complex structure $\tilde{J}$ on V with $\tilde{J}|_{V_k} = J_k$, and it follows that ϕ is $\tilde{J}$-convex. Define a homotopy of almost complex structures J^t, $t \in [0,1]$, on V as the concatenation of the homotopies J_k^t, $k \in \mathbb{N}$, carried out over the successively shorter time intervals $[1-2^{1-k}, 1-2^{-k}]$. Continuity of J^t for $t < 1$ follows from $J_{k-1}^1 = J_k^0$. Continuity at $t = 1$ holds because $J^t|_{V_k} = J_k$ for all $t \geq 1 - 2^{-k}$, so near every point J^t becomes independent of t for t close to 1. In particular, we have $J^1 = \tilde{J}$ and $J^0 = J_1^0 = J_0 = J$. This concludes the proof of Theorem 9.5. □

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