Math 53H Homework 8 Solutions

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2.) The first system has its only equilibrium point at $x = y = 0$. We are going to demonstrate that this equilibrium is not stable in the Lyapunov sense, and therefore also unstable in the asymptotic sense. Suppose for contradiction that the system is Lyapunov stable. Then there exists a $\delta > 0$ such that if $|x(0)| \leq \delta$ and $|y(0)| \leq \delta$, then the solution $(x(t), y(t))$ exists for all time and satisfies $|x(t)| < 1$, and $|y(t)| < 1$. Consider the solution with initial conditions $x(0) = y(0) = -\delta$. We can easily prove that $x(t) < 0$ and $y(t) < 0$ for all $t > 0$. Indeed, otherwise there exists a minimal $t > 0$ for which $x(t) = 0$ or $y(t) = 0$. Arguing with the mean value theorem we contradict the minimality of t. Since $x(t) < 0$ and $y(t) < 0$ for all t, we have $x(t)$ and $y(t)$ are decreasing for all t, in particular, $x(t) \leq -\delta$ and $y(t) \leq -\delta$ for all t. But then $\dot{x}(t) \leq -\delta$ and $\dot{y}(t) \leq -\delta^2$, so that in time at most δ^{-1}, $x(t) < -1$, a contradiction.

The second system has equilibria on the coordinate axes, whereever $x = 0$ or $y = 0$. All of these equilibria are Lyapunov unstable as we will show. For this system, we find that $y\dot{x} + x\dot{y} = 0$, so xy is a first integral for the system, i.e. $y = \frac{c}{x}$. Substituting this into the differential equation yields $\dot{x} = cx$, so $x = e^{ct}x_0$ and $y = e^{-ct}y_0$. Thus if both x_0 and y_0 are non-zero, then either $x \to \infty$ if $x_0y_0 > 0$ or $y \to \infty$ if $x_0y_0 < 0$, so the solution eventually escapes any neighborhood of any equilibrium.

3.) Motivated by the symmetry of the equations, let

$$y_1 = \frac{1}{2}(x_1 + x_2 + x_3 + x_4), \quad y_2 = \frac{1}{2}(x_1 + x_2 - x_3 - x_4),$$

$$y_3 = \frac{1}{2}(x_1 - x_2 - x_3 + x_4), \quad y_4 = \frac{1}{2}(x_1 - x_2 + x_3 - x_4).$$

In these coordinates we find that the differential equations take the form

$$\ddot{y}_1 = 0, \quad \ddot{y}_2 = -\frac{2k}{m}y_2, \quad \ddot{y}_3 = -\frac{2k}{m}y_3, \quad \ddot{y}_4 = -\frac{4k}{m}y_4$$

Thus

$$y_1 = C + C_1 t, \qquad y_2 = C_2\cos(\sqrt{\frac{2k}{m}}t + \theta_2),$$

$$y_3 = C_3\cos(\sqrt{\frac{2k}{m}}t + \theta_3) \qquad y_4 = C_4\cos(\sqrt{\frac{4k}{m}}t + \theta_4).$$

These are the principle oscillations, with frequencies $\sqrt{\frac{2k}{m}}$ and $\sqrt{\frac{4k}{m}}$.

In terms of the original coordinates, we have

$$x_1 = \frac{1}{2}\left[C + C_1 t + C_2\cos(\sqrt{\frac{2k}{m}}t + \theta_2) + C_3\cos(\sqrt{\frac{2k}{m}}t + \theta_3) + C_4\cos(\sqrt{\frac{4k}{m}}t + \theta_4)\right]$$

$$x_2 = \frac{1}{2}\left[C + C_1 t + C_2\cos(\sqrt{\frac{2k}{m}}t + \theta_2) - C_3\cos(\sqrt{\frac{2k}{m}}t + \theta_3) - C_4\cos(\sqrt{\frac{4k}{m}}t + \theta_4)\right]$$

$$x_3 = \frac{1}{2}\left[C + C_1 t - C_2\cos(\sqrt{\frac{2k}{m}}t + \theta_2) - C_3\cos(\sqrt{\frac{2k}{m}}t + \theta_3) + C_4\cos(\sqrt{\frac{4k}{m}}t + \theta_4)\right]$$

$$x_4 = \frac{1}{2}\left[C + C_1 t - C_2\cos(\sqrt{\frac{2k}{m}}t + \theta_2) + C_3\cos(\sqrt{\frac{2k}{m}}t + \theta_3) - C_4\cos(\sqrt{\frac{4k}{m}}t + \theta_4)\right]$$

Physically, we can describe these results as follows. The equations permit a linear drift of the entire system up or down the poles. At the middle frequency, masses on diagonally opposite rods (i.e. x_1 and x_3 or x_2 and x_4) oscillate with the same amplitude, but are always moving in the opposite direction. At the high frequency, all four masses move with the same amplitude, diagonally opposite masses moving in unison, and adjacent masses moving in opposite directions.

4.) The differential equation gives

$$(\ddot{x}_1, \ddot{x}_2) = (-u'(x_1 - x_2), u'(x_1 - x_2))$$

and so $\dot{x}_1 + \dot{x}_2$ is constant, and thus an integral curve of the system. This is different from the energy curve $\frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + U(x)$.

5.) Let $\psi_t(x) = \frac{\partial \phi_t(x)}{\partial t}$. Then $\psi_t(x)$ satisfies the differential equation

$$\frac{\partial}{\partial x}\psi_t(x) = D_2 f(x, \phi_t(x))\psi_t(x), \qquad \psi_t(0) = 1.$$

For each fixed t the above equation has solution

$$\psi_t(x) = e^{\int_0^x D_2 f(y, \phi_t(y))dy} > 0$$

since $\psi_t(0) = 1$.