Math 53H Homework 7 Solutions

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1.) The homogeneous equation has general solution $A\cos t + B\sin t$. We find a particular solution of $\ddot{x} + x = 5te^{-2t}$ by inverting $I + d^2$ on the vector space spanned by $f_1 = te^{-2t}$ and $f_2 = e^{-2t}$. With respect to this basis,

$$I + d^2 = \left[\begin{array}{cc} 5 & 0 \\ -4 & 5 \end{array}\right] \qquad \Rightarrow \qquad (I + d^2)^{-1} = \left[\begin{array}{cc} \frac{1}{5} & 0 \\ \frac{4}{25} & \frac{1}{5} \end{array}\right]$$

so that $(I + d^2)^{-1} 5 f_1 = f_1 + \frac{4}{5} f_2 = te^{-2t} + \frac{4}{5}e^{-2t}$.

Obviously $I + d^2$ is the zero operator on the span of $\cos t$ and $\sin t$, so we search for a particular solution of $\ddot{x} + x = 4\sin t$ in the space spanned by $t\cos t$ and $t\sin t$. Checking each, we find that $-2t\cos t$ is a solution. Thus the general solution is given by

$$x(t) = A\cos t + B\sin t + te^{-2t} + \frac{4}{5}e^{-2t} - 2t\cos t.$$

2.) The general homogeneous solution is $Ae^t \cos 2t + Be^t \sin 2t$.

To find particular solutions, we consider the operator $5I - 2d + d^2$ acting on spaces of quasi-polynomials. To find the particular solution for $ae^t \cos 2t$, we note $5I - 2d + d^2$ kills the span of $e^t \cos 2t$ and $e^t \sin 2t$, so we search for a solution in the space spanned by $te^t \cos 2t$ and $te^t \sin 2t$. Calculating the action on each of these, we find the particular solution $\frac{a}{10} te^t \cos 2t + \frac{a}{5} te^t \sin 2t$.

Searching for a particular solution for $-17 \sin 2t$ in the span of $\sin 2t$ and $\cos 2t$, we find the particular solution $-4 \cos 2t - \sin 2t$.

Thus the general solution is

$$x(t) = Ae^t \cos 2t + Be^t \sin 2t + \frac{a}{10} te^t \cos 2t + \frac{a}{5} te^t \sin 2t - 4 \cos 2t - \sin 2t.$$

For this to remain bounded as $t \to \infty$ we must have $a = 0$. Then the initial conditions given

$$A = 4 + \alpha, \qquad B = \frac{\beta}{2} - \frac{\alpha}{2} - 1.$$

Again the solution remains bounded only if $A = B = 0$. Thus $\alpha = -4$ and $\beta = -2$.

3.) The matrix of the homogenous equation is

$$\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -1 \end{bmatrix}$$

and so a fundamental system of solutions for the homogeneous equation is given by

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 2te^t + e^t \\ 2te^t \end{bmatrix} + c_2 \begin{bmatrix} -2te^t \\ e^t - 2te^t \end{bmatrix}.$$

Therefore, we seek a solution of the form

$$x(t) = 2c_1(t)te^t + c_1(t)e^t - 2c_2(t)te^t, \qquad y(t) = 2c_1(t)te^t + c_2(t)e^t - 2c_2(t)te^t.$$

Substituting this into the differential equation and canceling terms, we are left with

$$\dot{c}_1(t)(2te^t + e^t) - 2\dot{c}_2(t)(e^t) = 0, \qquad 2\dot{c}_1(t)(te^t) + \dot{c}_2(t)e^t(1-2t) = 15e^t\sqrt{t},$$

so that

$$\dot{c}_2(t) = (t + \frac{1}{2})\dot{c}_1(t), \qquad \dot{c}_1(t) = \frac{30\sqrt{t}}{1 + 4t - 4t^2}$$

Thus

$$c_1(t) = C + \int_0^t \frac{30\sqrt{x}dx}{1 + 4x - 4x^2}, \qquad c_2(t) = D + \int_0^t \frac{(30x^{\frac{3}{2}} + 15x^{\frac{1}{2}})dx}{1 + 4x - 4x^2}$$

where C and D are arbitrary constants.

4. The system is 2π periodic in both variables, so we will consider only values of $(x, y) \in [0, 2\pi) \times [0, 2\pi)$. Then there are 4 equilibrium points, at $x = 0$ or π and $y = 0$ or π. The four matrices of the linearized systems are

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \ (x = 0, y = 0), \qquad \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}, \ (x = \pi, y = 0),$$

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \ (x = 0, y = \pi), \qquad \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \ (x = \pi, y = \pi).$$

At $(0, 0)$ the eigenvalues are $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, so that the equilibrium is unstable. At $(\pi, 0)$ the eigenvalues are $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, so again, unstable. At $(0, \pi)$ the eigenvalues are $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, unstable. At (π, π) the eigenvalues are $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, so this point is stable (asymptotically).

5. The linearized system has matrix

$$\begin{bmatrix} a & -2 \\ 1 & 1 \end{bmatrix}$$

with characteristic polynomial $\lambda^2 - (1+a)\lambda + (a+2)$, and eigenvalues

$$\frac{(1+a) \pm \sqrt{(1+a)^2 - 4(a+2)}}{2}.$$

For $a < -2$ the radicand is real, and larger than $|1+a|$, so that there is a positive eigenvalue, and the system is unstable. For $a > -1$, the midpoint between the two eigenvalues is at $(1+a)/2 > 0$, so at least one eigenvalue has positive real root, and the system is unstable. For $-2 < a < -1$ both roots are negative and the system is asymptotically stable.

When $a = -2$ we can see in a relatively simple way that the system is unstable. Consider the ray $(x, y) = (z, -z)$ where $z > 0$. On this ray, $\dot{x} = z^2 = -\dot{y}$, so that this is an integral curve. Moreover, this integral curve diverges from zero as $t \to \infty$ so the system is Lyapunov unstable.

When $a = -1$ one gets the system

$$\begin{aligned} \dot{x} &= -x - 2y + x^2 \\ \dot{y} &= x + y + xy \end{aligned}$$

The eigenvalues of the linearized system are equal $\pm i$. Finding the canonical basis one gets the the change of variables

$$\begin{aligned} x &= u - v \\ y &= v \end{aligned}$$

transforms the system into

$$\begin{aligned} \dot{u} &= -v + (u - v)u \\ \dot{v} &= u + (u - v)v. \end{aligned}$$

It is useful to rewrite it in polar coordinates $u = r\cos\phi, v = r\sin\phi$.

Multipying first equation by u and second by v and adding them we get

$$2r\dot{r} = u\dot{u} + v\dot{v} = (u^2 + v^2)(u - v) = r^3(\cos\phi - \sin\phi),$$

or

$$\dot{r} = \frac{r^2}{2}(\cos\phi - \sin\phi). \tag{1}$$

Similarly, Multipying first equation by v and second by u and subtracting second from the first one we get

$$\dot{u}v - \dot{v}u = -(u^2 + v^2),$$

and plugging $u = r\cos\phi, v = r\sin\phi$ we get $\dot{\phi} = -1$. Hence, $\phi = -t$ (up to choice of the initial time) and (1) takes the form

$$\frac{dr}{d\phi} = \frac{r^2}{2}(\sin\phi - \cos\phi).$$

This equation with seprable variables which is easy to solve and we get that all trajectories in some neighborhood of the origin are periodic and all phase curves there are closed curve surrounding the equilibrium point. Hence the system is Lyapunov stable but asymptotically unstable.

6. If $\omega = \pm\nu$ then the systems are the same after possibly exchanging the names of the variables. If $\omega \neq \pm\nu$ then the systems cannot be topologically equivalent. This is because solutions of the first system have period $2\pi/|\omega|$ while solutions of the second system have period $2\pi/|\nu|$. A homeomorphism giving topological equivalence maps periodic solutions to periodic solutions of the same period (check this!), which proves the inequivalence.