Math 53H Homework 6 Solutions

May 24, 2012

1.) Let the eigenvalues of A be $\lambda_1, \lambda_2, ..., \lambda_k$, with root spaces of dimension $d_1, d_2, ..., d_k$. By the general theory, each coordinate function in $x(t)$ satisfying $\dot{x}(t) = Ax(t)$ may be expressed as a linear combination of form

$$x_i(t) = \sum_{j=1}^{k} P_{i,j}(t) e^{\lambda_j t} \tag{1}$$

where $P_{i,j}(t)$ is a polynomial of degree at most $d_j - 1$. In particular, $t^2 + t \sin t$ is of this form.

Let $L = \max\{\Re(\lambda_j) : 1 \leq j \leq k\}$ and let s be an arbitrary complex number with real part greater than L. Then we define the *Laplace Transform* of x_i at complex argument s by

$$\mathcal{L}(x_i)(s) = \int_0^\infty x_i(t) e^{-st} dt.$$

Evidently $\mathcal{L}(x_i)(s)$ is a linear combination of terms of form $\int_0^\infty P_{i,j}(t) e^{-(s-\lambda_j)t} dt$. Let $P(t) = \sum_{\ell=0}^m a_\ell t^\ell$. Then for $\Re(s) > \Re(\lambda)$ we have

$$\int_0^\infty P(t) e^{-(s-\lambda)t} dt = \sum_{\ell=0}^m a_\ell \int_0^\infty t^\ell e^{-(s-\lambda)t} dt = \sum_{\ell=0}^m \frac{a_\ell \ell!}{(s-\lambda)^{\ell+1}}. \tag{2}$$

Notice that while the integral made sense only for $\Re(s) > \Re(\lambda)$, the right hand side of (2) makes sense for any $s \neq \lambda$, and the degree of the polynomial $P(t)$ is characterized by the number of powers of $(s-\lambda)$ required to be multiplied times $\mathcal{L}(P(t)e^{\lambda t})(s)$ so that the function has a finite limit as $s \to \lambda$. In this way, we interpret $\mathcal{L}(x_i)(s)$ as defined for all $s \neq \lambda_i$ for any i, and we see that we may read off the eigenvalues and polynomials appearing in $x_i(t)$ from the rate of growth of $\mathcal{L}(x_i)(s)$ as s tends to various λ_i.

In the case of $x_1(t) = t^2 + t \sin t$, we find

$$\mathcal{L}(x_1)(s) = \frac{2}{s^3} + \frac{2s}{(s^2+1)^2} = \frac{2}{s^3} + \frac{1}{(s+i)^2} + \frac{1}{(s-i)^2}.$$

In particular, the root space at 0 has dimension at least 3, and each root space at $\pm i$ has dimension at least 2, so A must be at least 7×7.

2.) The homogenous solution has form $c_1e^t + c_2e^{-t}$. Indeed, it is readily checked that both e^t and e^{-t} satisfy the homogeneous equation, and since the equation is degree 2, these are all solutions, by the general theory. A particular solution is $x_p(t) = -1$. Therefore, the complete solution is $c_1e^t + c_2e^{-t} - 1$. Boundedness implies $c_1 = 0$, and then the initial condition implies $c_2 = 1$ so $e^{-t} - 1$ is the desired solution.

3.) Answer: $a : a = ((2n+1)\pi)^2$, $n \in \mathbb{Z}$.
Proof: If $a = 0$ then a solution is given by $t^2/2 - t/2$.
If $a = -\lambda^2 < 0$ then the general form of the solution is

$$c_1 e^{-\lambda t} + c_2 e^{\lambda t} - 1/\lambda^2.$$

The initial condition is equivalent to

$$c_1 + c_2 = 1/\lambda^2, \qquad c_1 e^{-\lambda} + c_2 e^{\lambda} = 1/\lambda^2,$$

and this has a solution for real c_1, c_2.

Suppose $a = \lambda^2 > 0$ and $\lambda \neq (2n+1)\pi$ for some $n \in \mathbb{Z}$. Observe that we may solve (for ξ) the equation

$$\sin(\lambda\xi) = \sin(\lambda(\xi + 1))$$

since the average value of $\sin(\lambda\xi) - \sin(\lambda(\xi+1))$ is zero as ξ ranges over $[0, 2\pi/\lambda]$. Let ξ_0 be such a solution and set $\zeta = \sin(\lambda\xi_0)$. We may find such a pair ξ_0, ζ with $\zeta \neq 0$. Actually, this follows from considering seperately two cases: if $\lambda = 2n\pi$ then $\sin(\lambda\xi) = \sin(\lambda(\xi+1))$ holds for all ξ, and we just choose a ξ_0 with $\zeta \neq 0$. If $\lambda \neq 2n\pi$ then λ is not an integer multiple of π, so that if $\sin(\lambda\xi) = 0$ then $\sin(\lambda(\xi+1)) \neq 0$ and $\zeta \neq 0$ is forced.

Having found ξ_0, $\zeta \neq 0$, a solution of the equation is given by

$$\frac{-1}{\zeta\lambda^2}\sin(\lambda(\xi_0 + t)) + \frac{1}{\lambda^2}.$$

It remains to consider the case $a = \lambda^2 = ((2n+1)\pi)^2$. In this case, observe that the general form of the homogenous solution is $y(t) = c_1 e^{i\lambda t} + c_2 e^{-i\lambda t} + \frac{1}{\lambda^2}$ so that $y(t+1) = -y(t)$ for all solutions of the homogeneous equation. A particular solution is $x_p(t) = \frac{1}{\lambda^2}$. Therefore the general solution has form

$$x(t) = \frac{1}{\lambda^2} + y(t)$$

where $y(t+1) = -y(t)$. Then for the initial conditions to hold we must have $y(0) = y(1)$ which forces $y(0) = y(1) = 0$, but this does not satisfy the initial conditions.

4.) The system may be rewritten $\dot{x} = Ax$ where $x = [x_1, x_2, x_3]^t$ (replacing x, y, z) and

$$A = \begin{bmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Evidently the root space at 2 for A is all of $\mathbb{R}^3$. We calculate

$$(A-2I)e_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix},$$

$$(A-2I)^2 e_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

$$(A-2I)^3 e_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so that

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{-1}.$$

The general solution with initial condition x_0 is then given by

$$x(t) = e^{At}x_0 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} & t^2/2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{-1} x_0.$$

5.) Recall that $\frac{d}{dt}e^{tA} = Ae^{tA}$. Differentiating both sides of the equality twice with respect to t and setting $t = 0$ we find $4A^2 = A^2$ so $A^2 = 0$. It follows that A is nilpotent, with nilpotent steps of length at most 1, so that the Jordan form of A is composed of blocks of type $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $[0]$.