Math 53H Homework 5 Solutions

May 24, 2012

1.) For fixed s we have

$$\begin{aligned}
\left.\frac{\partial}{\partial t} e^{As}e^{Bt}e^{-As}e^{-Bt}\right|_{t=0} &= \lim_{t\to 0}\frac{1}{t}\left[e^{As}(1+Bt+O(t^2))e^{-As}(1-Bt+O(t^2))-1\right] \\
&= \lim_{t\to 0}\frac{1}{t}\left[\left(e^{As}+e^{As}Bt+O(t^2)\right)\left(e^{-As}-e^{-As}Bt+O(t^2)\right)-1\right] \\
&= \lim_{t\to 0}\frac{1}{t}\left[-Bt+e^{As}Bte^{-As}+O(t^2)\right] = -B+e^{As}Be^{-As}
\end{aligned}$$

Thus

$$\begin{aligned}
\left.\frac{\partial^2}{\partial s\partial t} e^{As}e^{Bt}e^{-As}e^{-Bt}\right|_{s=0} &= \lim_{s\to 0}\frac{1}{s}\left[-B+(1+As+O(s^2))B(1-As+O(s^2))\right] \\
&= \lim_{s\to 0}\frac{1}{s}\left[ABs-BAs+O(s^2)\right] \\
&= AB-BA.
\end{aligned}$$

2.) Evidently this holds if A is a multiple of the identity matrix. We show that this is the only possibility.

Matrix A has Jordan normal form, so for an appropriate choice of C we have $C^{-1}AC = A = J$ with J in Jordan normal form, so we may assume that A is in Jordan normal form.

Let A have eigenvalues $A_{i_1,i_1} = \lambda_1$ and $A_{i_2,i_2} = \lambda_2$ at positions i_1 and i_2 on the diagonal, and let S be the matrix with $S_{i_1,i_2} = S_{i_2,i_1} = 1$, 0's everywhere else off the diagonal, and $S_{i_1,i_1} = S_{i_2,i_2} = 0$, $S_{j,j} = 1$ for $j \neq i_1, i_2$. Then $B = S^{-1}AS$ has $B_{i_1,i_1} = \lambda_2$. But $B = A$ so $\lambda_1 = \lambda_2$ and all eigenvalues of A are equal.

Now suppose A has a 1 off the diagonal, say $A_{i,i+1} = 1$. Let S be the matrix with $S_{i,i+1} = S_{i+1,i} = 1$, all other non-diagonal entries 0, and $S_{i,i} = S_{i+1,i+1} = 0$, $S_{j,j} = 1$ for $j \neq i, i+1$. Then $A = S^{-1}AS$ has $A_{i+1,i} = 1$, which is below the diagonal, a contradiction. It follows that A has only diagonal entries, all of the same value, hence is a scalar multiple of the identity.

3.) We have $\chi_{A^{-1}}(\lambda) = \det(A^{-1} - \lambda I)$. By the multplicative property of the determinant, it follows that

$$\begin{aligned}
\chi_{A^{-1}}(\lambda) &= \det(A)^{-1} \det(I - \lambda A) \\
&= (-\lambda)^n \det(A)^{-1} \det(A - \lambda^{-1} I) \\
&= (-1)^n \det(A)^{-1} \lambda^n \chi_A(\lambda^{-1}).
\end{aligned}$$

4.) With the standard basis $e_1, e_2, ..., e_m$ we have $J_m e_i = e_{i+1}$ for $i < m$, and $J_m e_m = 0$. Therefore, $J_m^2 e_i = e_{i+2}$ for $i \leq m-2$ and $J_m^2 e_{m-1} = J_m^2 e_m = 0$. Thus the matrix of J_m^2 with respect to basis $e_1, e_3, ..., e_{2\lceil m/2 \rceil - 1}, e_2, e_4, ..., e_{2\lfloor m/2 \rfloor}$ is composed of two Jordan blocks with 0's on the diagonal, the first having length $\lceil m/2 \rceil$ and the second having length $\lfloor m/2 \rfloor$.

5.) The characteristic polynomial is $\lambda^2(2-\lambda)^2$. We calculate the null space of A as the span of $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Since the root space is 2-dimensional, we need to find v_2 such that $Av_2 = v_1$. Solving this affine linear system, we find that $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ will do.

We calculate the null space of $A - 2I$ as the span of $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Solving $(A-2I)v_4 = v_3$ we find that $v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ will do. Thus

$$A[v_1, v_2, v_3, v_4] = [v_1, v_2, v_3, v_4] \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Thus $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ and $C = [v_1, v_2, v_3, v_4] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

[Having misordered the vectors in the above matrices more than once, I include the following reminder on how to multiply matrices : the matrix multiplication

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = av_1 + bv_2 + cv_3 + dv_4$$

is familiar to everyone as the dot product if v_1, v_2, v_3, v_4 are scalars. It is still true if v_1, v_2, v_3, v_4 are vectors. Thus the above calculation says, e.g. that A acts on v_3 by multiplying it by 2, and it acts on v_4 by multiplying it by 2 and adding one copy of v_3.]