Math 53H Homework 3 Solutions

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1. The equation is homogeneous. Making change of variables $(x, t) = (ue^s, e^s)$ we obtain the equation with initial conditions

$$ds = \cot u du, \qquad u(0) = \frac{\pi}{6}$$

which has solution $s = \ln 2 \sin u$, or $t = 2\sin(\frac{x}{t})$. We deduce $x = t \arcsin \frac{t}{2}$ for $0 < t < 2$. As $t \to 2$, $\frac{x}{t} \to \frac{\pi}{2}$ and, therefore, $\dot{x} \to \infty$ (i.e. the defining equation is no longer defined), so this is a boundary for extension of the solution.

2. The equation for x is linear inhomogeneous, and hence has general solution

$$x(t) = ce^{-t} + \int_{t_0}^{t} e^{\xi - t} f(\xi) d\xi.$$

Suppose that this solution is bounded. In particular

$$e^{-t}\left[c + \int_{t_0}^{t} e^{\xi} f(\xi) d\xi\right]$$

remains bounded as $t \to -\infty$, i.e.

$$\lim_{t \to -\infty} c + \int_{t_0}^{t} e^{\xi} f(\xi) d\xi = 0.$$

Notice that the limit in the integral certainly exists, since f is bounded. We deduce that $c = \int_{t_0}^{-\infty} e^{\xi} f(\xi) d\xi$, and, therefore, if a bounded solution exists, it must equal

$$x(t) = \int_{-\infty}^{t} e^{\xi - t} f(\xi) d\xi.$$

This solution is indeed bounded, since

$$|x(t)| \leq M \int_{-\infty}^{t} e^{\xi - t} d\xi = M \int_{-\infty}^{0} e^{-\xi} d\xi = M.$$

If f is periodic with period T, then by making a change of variables in the integral

$$x(t+T) = \int_{-\infty}^{T+t} e^{\xi - t - T} f(\xi) d\xi = \int_{-\infty}^{t} e^{\zeta - t} f(\zeta - T) d\zeta = \int_{-\infty}^{t} e^{\zeta - t} f(\zeta) d\zeta = x(t),$$

and so x is also periodic.

3. Since the equation is linear inhomogeneous, the general solution is given by

$$x(t) = ce^{t+\frac{1}{2}\sin 2t} - e^{t+\frac{1}{2}\sin 2t} \int_0^t e^{-\xi-\frac{1}{2}\sin 2\xi} \sin \xi d\xi.$$

If this function is periodic, then it is bounded, and therefore

$$e^{t+\frac{1}{2}\sin 2t} \left[c - \int_{t_0}^t e^{-\xi-\frac{1}{2}\sin 2\xi} \sin \xi d\xi \right]$$

remains bounded as $t \to \infty$, i.e.

$$c = \int_{t_0}^\infty e^{-\xi-\frac{1}{2}\sin 2\xi} \sin \xi d\xi.$$

Thus, if a periodic solution exists, it is equal to

$$x(t) = e^{t+\frac{1}{2}\sin 2t} \int_t^\infty e^{-\xi-\frac{1}{2}\sin 2\xi} \sin \xi d\xi.$$

We have

$$x(t+2\pi) = e^{t+2\pi+\frac{1}{2}\sin 2t} \int_{t+2\pi}^\infty e^{-\xi-\frac{1}{2}\sin 2\xi} \sin \xi d\xi = e^{t+\frac{1}{2}\sin 2t} \int_t^\infty e^{-\xi-\frac{1}{2}\sin 2\xi} \sin \xi d\xi = x(t),$$

and so x is 2π-periodic.

4. Since the solution of the differential equation depends smoothly on the initial conditions, we may differentiate the differential equation with respect to u to deduce that

$$\frac{\partial^2}{\partial t \partial u}\phi_u(t) = \frac{\partial}{\partial u}\phi_u(t) + 2\phi_u(t)\frac{\partial}{\partial u}\phi_u(t) + 3t\phi_u(t)^2\frac{\partial}{\partial u}\phi_u(t).$$

Since $\phi_0 \equiv 0$, evaluating at $u = 0$, and writing $\psi(t) = \frac{\partial}{\partial u}\phi_u(t)\big|_{u=0}$, we deduce that ψ satisfies differential equation with initial condition

$$\psi'(t) = \psi(t), \qquad \psi(2) = 1.$$

Thus $\psi(t) = e^{t-2}$.

5. a) Before we begin we recall the following identity. Let v be a vector field on space X and α a differential 1-form on space Y, with diffeomorphism $f : X \to Y$. Then, writing subscripts to indicate where various functions are evaluated,

$$(f^*\alpha)_x(v_x) = \alpha_{f(x)}\left(Df_x v_x\right) = \alpha_{f(x)}\left(f_* v_{f(x)}\right),$$

by the definition of pull-back and push-forward.

Let $\alpha = dz - ydx$. Let $v(x, y, z)$ be any vector field contained in the plane field ξ. Consider

$$F(x,t) = \alpha_{Z^t(x)}(Z^t_* v_{Z^t(x)}) = ((Z^t)^*\alpha)_x(v_x).$$

Observe that by Cartan's formula

$$\begin{aligned} L_Z\alpha &= d(Z\neg\alpha) + Z\neg d\alpha \\ &= d(z - xy) + Z\neg(dx \otimes dy - dy \otimes dx) \\ &= dz - xdy - ydx + 2ydx - xdy = dz - ydx = \alpha. \end{aligned}$$

Therefore

$$\frac{d}{dt}(Z^t)^*\alpha\bigg|_{t=t_0} = (Z^{t_0})^* L_Z\alpha = (Z^{t_0})^*\alpha.$$

We may write $(Z^t)^*\alpha = \alpha_1(t)dx + \alpha_2(t)dy + \alpha_3(t)dz$, and each α_i satisfies $\dot{\alpha}_i = \alpha_i$, so that in fact $(Z^t)^*\alpha = e^t\alpha$. It then follows that $F(x,t) = e^t F(x,0) = 0$ for all t, so for all t, $Z^t_* v$ remains in ξ. Since Z^t is a diffeomorphism, dZ^t has full rank, and in particular $Z^t_*\xi$ remains two dimensional for all t. Thus we have shown that $Z^t_*\xi = \xi$.

b) Let

$$v_1(x,y,z) = \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}, \qquad v_2(x,y,z) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

be a basis of vector fields for ξ. Suppose for contradiction that $f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ is a diffeomorphism $f : U \to W$, and $f_*\xi = \{dz = 0\}$. This is equivalent to

$$dz(f_* v_1) = \frac{\partial f_3}{\partial x} + y\frac{\partial f_3}{\partial z} = 0, \qquad dz(f_* v_2) = \frac{\partial f_3}{\partial y} = 0.$$

In particular, the second condition says that f_3 is a function of x and z only. The same is therefore true for $\frac{\partial f_3}{\partial x}$ and $\frac{\partial f_3}{\partial z}$ and so for each fixed x and z in U, $\frac{\partial f_3}{\partial x} + y\frac{\partial f_3}{\partial z}$ is a linear function of y which vanishes identically. This is only possible if $\frac{\partial f_3}{\partial x} = \frac{\partial f_3}{\partial z} = 0$. But then Df has determinant 0 and cannot be a diffeomorphism.