

## Math 53H Homework 2 Solutions

April 23, 2012

1.4.4 8) We will identify the real vector space of polynomials of degree at most  $n$  with  $\mathbb{R}^{n+1}$  via

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \quad \mapsto \quad [a_n, a_{n-1}, \dots, a_0].$$

Any other choice of the identification is equivalent up to a change of basis.

We first check that  $\{g^t : t \in \mathbb{R}\}$  is a one-parameter group of linear transformations. Given scalars  $a$  and  $b$  and polynomials  $p$  and  $q$

$$g^t(ap + bq)(x) = (ap + bq)(x + t) = ap(x + t) + bq(x + t),$$

and so

$$g^t(ap + bq) = ag^t(p) + bg^t(q),$$

which proves the linearity for each  $t$ .

We have

$$g^{t_1}(g^{t_2}(p))(x) = (g^{t_2}(p))(x + t_1) = p(x + t_1 + t_2) = g^{t_1+t_2}(p)(x),$$

which proves the group law. Since  $g^0(p) = p$ ,  $g^t$  has inverse  $g^{-t}$ . This proves that  $g^t$  is a one-parameter group. Furthermore, as an invertible linear map on  $\mathbb{R}^{n+1}$ ,  $g^t$  is automatically a diffeomorphism.

It remains to show that  $g^t(p)$  is smooth as a function of  $t$  and of the argument  $p$ . Since  $g^t$  is a linear map of its polynomial argument, it is automatically smooth. With respect to the basis identifying polynomials with  $\mathbb{R}^{n+1}$ , the map  $g^t$  corresponds to the matrix multiplication

$$\begin{bmatrix} \binom{n}{0} & 0 & 0 & \dots & 0 \\ \binom{n}{1}t & \binom{n-1}{0} & 0 & \dots & 0 \\ \binom{n}{2}t^2 & \binom{n-1}{1}t & \binom{n-2}{0} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{n}t^n & \binom{n-1}{n-1}t^{n-1} & \binom{n-2}{n-2}t^{n-2} & \dots & \binom{0}{0} \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_0 \end{bmatrix}.$$

Since this matrix has entries that are polynomials in  $t$ , it is smooth as a function of  $t$ .

The velocity vector field of  $g^t$  is found by differentiating  $g^t$  at  $t = 0$ . At argument  $p$  and for any  $x$  this yields

$$\left. \frac{d}{dt}(g^t p) \right|_{t=0} = \lim_{t \rightarrow 0} \frac{p(x+t) - p(x)}{t} = p'(x).$$

Thus the velocity vector at  $p$  is given by  $p'$ .

1.5.2 2) Let  $f(x) = e^x$ . We wish to calculate  $f_*v(y), y > 0$  for the vector field  $x \frac{\partial}{\partial x}$ . Let  $\phi(t) = x + tx$  with  $x = \ln y$ ; at  $t = 0$  this is a curve passing through  $x = f^{-1}(y)$  with velocity equal to  $v(x)$ . Thus

$$f_*v(y) = \left. \frac{d}{dt}(f \circ \phi) \right|_{t=0} = \frac{d}{dt} e^{(\ln y + t \ln y)} = y \ln y.$$

The velocity is positive (there are only two possibilities of direction on  $\mathbb{R}$ ), so the field is given by  $f_*v(y) = y \ln y \frac{\partial}{\partial y}$ .

1.6.5 3) Note that the quasi-homogenous assumption on  $v = \sum_{i=1}^n v_i(x) \frac{\partial}{\partial x_i}$  translates to  $v_i$  being quasi-homogeneous of degree  $r + \alpha_i$  for each  $i$ .

Let  $\Gamma : [0, T] \rightarrow \gamma$  solve  $\frac{d\Gamma}{dt} = v(\Gamma(t))$ , that is,  $\Gamma$  is the solution of the differential equation on the closed curve  $\gamma$ . Notice that  $g^s$  is a diffeomorphism, and therefore  $g^s \circ \Gamma$  is still a smooth curve, and so it satisfies  $g^s \circ \Gamma(T) = g^s \circ \Gamma(0)$ , and for  $0 < t < T$ ,  $g^s \circ \Gamma(t) \neq g^s \circ \Gamma(0)$ , so that  $g^s \circ \gamma$  is again a closed curve, and  $g^s \circ \Gamma$  still has period  $T$ .

Recall that  $g^s$  is dilation by  $e^{\alpha_i s}$  in the  $i$ th direction. Therefore, for fixed  $s$  its derivative is the diagonal matrix  $\text{diag}[e^{\alpha_1 s}, e^{\alpha_2 s}, \dots, e^{\alpha_n s}]$ . In particular,

$$\frac{d}{dt}(g^s \Gamma(t)) = \text{diag}[e^{\alpha_1 s}, e^{\alpha_2 s}, \dots, e^{\alpha_n s}] \Gamma'(t) = \text{diag}[e^{\alpha_1 s}, e^{\alpha_2 s}, \dots, e^{\alpha_n s}] v(\Gamma(t)).$$

But the quasi-homogeneity condition on  $v$  gives that

$$v(g^s \Gamma(t)) = e^{rs} \text{diag}[e^{\alpha_1 s}, e^{\alpha_2 s}, \dots, e^{\alpha_n s}] v(\Gamma(t)).$$

Thus  $t \mapsto g^s(\Gamma(t))$  has derivative  $e^{-rs} v(\Gamma(t))$ . It follows that  $t \mapsto H(t) = (g^s(\Gamma(e^{rs}t)))$  satisfies  $H'(t) = v(H(t))$ , and  $H$  traverses  $g^s \gamma$  in time  $e^{-rs}T$ . This proves that  $g^s(\gamma)$  is a phase curve, of period  $e^{-rs}T$ .

4) a. In polar coordinates, the first differential equation is given by

$$r' = 0, \quad \theta' = r^2$$

while the second equation is given by

$$r' = 0, \quad \theta' = 1.$$

Thus solutions in polar coordinates are given by  $r = c_1, \theta = c_2 + c_1^2 t$  for the first equation, and  $r = c_1, \theta = c_2 + t$  for the second.

b. The explicit solutions above show that all phase curves in both systems are periodic. In the first system, the period of phase at radius  $r$  is  $\frac{2\pi}{r^2}$ , while in the second system, the period of all phases is 1. A diffeomorphism of vector fields carries periodic solutions to periodic solutions of the same period. It follows that the two vector fields cannot be diffeomorphic.

c. The identity is a diffeomorphism of the line curves since  $x'_1 : x'_2$  are in the same proportion in each field.

5) Let  $x_1 = -x'$ . Then this system is the same as the second system of problem 4. It follows that  $x(t) = r_1 \cos(c_1 + t)$  for some constants  $r_1, c_1$ . In the system  $y''(t) = -a^2 y(t)$ , setting  $at = s$  we find  $\frac{d^2}{ds^2} y(s) = -y(s)$ , so  $y(t) = r_2 \cos(c_2 + at)$  for some constants  $r_2$  and  $c_2$ . The periods of  $x$  are at  $2\pi n, n \in \mathbb{Z}$ , unless  $r_1 = 0$  in which case  $x$  is constant. The periods of  $y$  are at  $2\pi m/a, m \in \mathbb{Z}$ , unless  $r_2 = 0$ , in which case  $y$  is constant. A period of the system must be a common period of  $x$  and  $y$ . The only solution to  $2\pi n = 2\pi m/a$  is  $n = m = 0$ , since  $a$  is irrational, so there are no periods of the system if both  $r_1$  and  $r_2$  are non-zero. If  $r_2 = 0$  but  $r_1 \neq 0$  the system has periods of length  $2\pi$ . If  $r_1 = 0$  but  $r_2 \neq 0$  the system has periods of length  $2\pi/a$ . If both  $r_1$  and  $r_2$  are zero, the system is constant.