Math 53H Homework 1 Solutions

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1.1.18 1) We have $\frac{d}{dt}(x_1^2 + x_2^2 + x_3^2 + x_4^2) = 0$, which proves that this quantity is constant.

2) Let position be given by $p = x_1(t)e_1 + x_2(t)e_2 + x_3(t)e_3 + x_4(t)e_4$ and velocity by $v = x_2(t)e_1 - x_1(t)e_2 + x_4(t)e_3 - x_3(t)e_4$. Let the dual basis to $e_1, ..., e_4$ be $y_1, ..., y_4$ and consider the dual one-forms

$$\hat{p} = x_1(t)y_1 + x_2(t)y_2 + x_3(t)y_3 + x_4(t)y_4, \qquad \hat{v} = x_2(t)y_1 - x_1(t)y_2 + x_4(t)y_3 - x_3(t)y_4.$$

Then $p \wedge v$ is a two-form. Its action on a pair of (u, w) is to give the signed area of $\text{Proj}(u), \text{Proj}(w)$ in the plane described by p and v, scaled by the area of the parallelopiped described by position and velocity. [Notice that the length of position and velocity are constants by 1), and they are orthogonal, so the scaling factor is a constant.]

We have $\frac{d}{dt}(p \wedge v) = v \wedge v - p \wedge p = 0$, so $(p \wedge v)$ does not change with time. In particular, this implies that the plane P described by position and velocity is fixed. In particular, for all time position lies on the intersection of the sphere with P. Since P is a plane passing through the origin, this intersection is a great circle.

1.1.26 1) Separating variables, $\frac{dy}{dx} = \frac{y}{x}$ leads to $\ln|y| = \ln|x| + C$ or $|y| = e^C|x|$ which is two lines. $\frac{dy}{dx} = \frac{x}{y}$ leads to $y^2 - x^2 = C$, which is a hyperbola with branches opening up/down if $C > 0$ and right/left if $C < 0$. If $C = 0$ this is intersecting lines again. If $\frac{dy}{dx} = -\frac{x}{y}$ we obtain $x^2 + y^2 = C$, a circle. If $\frac{dy}{dx} = -\frac{y}{x}$ then we obtain $\ln|y| = -\ln|x| + C$ so $|y| = \frac{e^C}{|x|}$, which is two hyperbolas with asymptotes on the axes.

1.1.27 1) In the region R defined by $y \leq k/a$ and $x \leq \ell/b$ we have x is increasing and y is decreasing, so that this regions alone does not contain a period. We may obviously assume that the initial point (x_0, y_0) lies in this region. For $(x,y) \in R$, $\frac{dx}{dt} < kx$ and $\frac{dy}{dt} < bxy \leq \frac{b^2}{\ell} y$. It follows that while $(x,y) \in R$, $x(t) \leq x_0 e^{kt}$ and $y(t) \leq y_0 e^{b^2 t/\ell}$ (we give a detailed proof of this below). Then the first exit time from R is bounded by $\min(\frac{\ell}{b^2} \log \frac{k/a}{y_0}, \frac{1}{k} \log \frac{\ell/b}{x_0})$, and this quantity tends to infinity as $(x_0, y_0) \to (0,0)$.

We now show that if there is an interval $[0,T]$ on which $x'(t) < kx(t)$ then also $x(t) \leq x(0)e^{kt}$ on $[0,T]$. We know that x is differentiable, and in fact C^∞, since successive derivatives are expressed in terms of lower order derivatives. At $t = 0$ $x'(0) < kx_0$ and therefore $x'(t) < kx_0$ continues to hold for $t < \delta$ for some sufficiently small δ. In particular, integrating x' we find that $x(t) \leq x_0 + ktx_0 < x_0 e^{kt}$ for $t \in (0, \delta)$, by the mean value theorem. Suppose for contradiction that there is some point $t_1 > 0$ such that $x(t_1) \geq x_0 e^{kt_1}$. If such a point exists then there is a first such point, since we may set

$$t_0 = \inf\{t > 0, x(t) \geq x_0 e^{kt}\}.$$

We know that $t_0 \geq \delta$, and $x(t_0) = 0$ by continuity of x, since $x < 0$ for $t < t_0$. By the mean value theorem, there is a $t \in (0, t_0)$ satisfying $x'(t) = kx_0 e^{kt}$. But $x'(t) \leq kx(t) < kx_0 e^{kt}$ since $t \in (0, t_0)$, a contradiction. The proof of the bound for $y(t)$ is similar.

4) The equation is equivalent to

$$e^{-t/2}dt = -\frac{dy}{y}, \qquad y(0) = 1$$

which has solution

$$\ln|y(t)| = 2(e^{-t/2} - 1).$$

Since $y(0) = 1$, y is continuous, and $\ln|y|$ is finite for finite t, we deduce that $y \neq 0$ and therefore y is positive for all time. Hence we obtain

$$y(t) = e^{2(e^{-t/2}-1)}.$$

1.3.2 2) Since $\phi(t) = t + c$, we may write the equation as

$$\frac{dr}{dt} = (r^2 - 1)(2r\cos(t + c) - 1).$$

Substituting $Y = r - 1$, $X = t$ we obtain

$$\frac{dY}{dX} = Y(Y + 2)(2(Y + 1)\cos(X + c) - 1),$$

which is an equation of the form considered in section 1.3.1. The corresponding linearized equation is

$$\frac{dY}{dX} = Y[4\cos(X + c) - 2].$$

The corresponding multiplier is $\ln\lambda = \int_0^{2\pi}[4\cos(X + c) - 2]dX = -4\pi$ so $\lambda < 1$. Thus $r = 1$ is stable by problem 1 of this section.

1.2.7 2) The phase equation is

$$bx - \ell \ln x + ay - k \ln y = C.$$

Since the initial point (x_0, ϵ) lies on this curve, we have

$$C = bx_0 - \ell \ln x_0 + a\epsilon - k \ln \epsilon.$$

In particular, as $\epsilon \to 0$ we have $C = -k \ln \epsilon + O(1)$ where $O(1)$ denotes a quantity that is bounded as $\epsilon \to 0$. Notice that, since C is constant, $x > 0$ and $y > 0$ for all time. For positive x and y, the expressions

$$bx - \ell \ln x, \qquad ay - k \ln y$$

each have a unique minimum, at $x = \frac{\ell}{b}$ and $y = \frac{k}{a}$ respectively. In particular, these expressions are bounded below by fixed constants independent of t and ϵ. We deduce that for all time

$$bx \leq -k \ln \epsilon + O(1), \qquad -\ell \ln x \leq -k \ln \epsilon + O(1)$$

and

$$ay \leq -k \ln \epsilon + O(1), \qquad -\ln y \leq -\ln \epsilon + O(1)$$

where each $O(1)$ denotes a quantity that is bounded for all t and all $\epsilon < 1$.

For each fixed x_1 there are either 0, 1 or 2 solutions to

$$ay - k \ln y = C - bx_1 + \ell \ln x_1 \tag{1}$$

depending on whether the right hand side is smaller, equal, or larger than the global minimum value of the left. Let x_m and x_M be the minimum and maximum values for x at which (1) has a single solution. By convexity of $bx - \ell \ln x$, there are two solutions $y_1(x) > y_2(x)$ of (1) for $x \in (x_m, x_M)$ and no solutions for $x \notin [x_m, x_M]$. The union of the graphs $x \mapsto (x, y_1(x))$, $x \mapsto (x, y_2(x))$ on the interval $[x_m, x_M]$ describe the phase curve C. Function $y_1(x)$ is concave and function $y_2(x)$ is convex, since their graphs bound the convex set defined by $f(x, y) \leq C$.

Now we turn to the problem at hand and give a lower bound for period length. Since the minimum value of $ay - k \ln y$ is a constant, say c_y, we have that x_m and x_M solve equation

$$bx - \ell \ln x = c_y + a\epsilon - k \ln \epsilon.$$

For all ϵ sufficiently small it follows that

$$x_M \geq \frac{-k}{2b} \ln \epsilon,$$

and also $\ln x_m \leq \frac{k}{\ell} \ln \epsilon + O(1)$. Now for all x on the curve $\frac{dx}{dt} = kx - axy \leq kx$. Therefore, if x is at x_m at time t_0 then for $t > t_0$ we have $x(t) \leq x_m e^{kt}$, i.e. by arguing as in problem 1.1.27 1). Therefore, if x reaches x_M at t_1 we must have

$$t_1 - t_0 \geq \frac{1}{k} \ln \frac{x_M}{x_m} \geq \frac{1}{\ell} \ln \epsilon + O(1).$$

Since $t_1 - t_0$ is a lower bound for the length of the period, the length of period grows at least as large as a constant times $-\ln \epsilon$ as $\epsilon \to 0$.

Now we prove the upper bound. Split the phase curve into segments where $y > 3k/2a$, where $y < k/2a$ and where $k/2a \le y \le 3k/2a$. By convexity, the first and second segments are connected, while the third segment is actually two disjoint segments. Consider first the top segment. Here

$$\frac{dx}{dt} = kx - axy < \frac{-k}{2}x.$$

Therefore, if $t_0 < t_1$ denote two times on this segment (without leaving the segment) then $x(t_1) \le e^{-k(t_1-t_0)/2}x(t_0)$. It follows that the total time spent on this segment is bounded by

$$\frac{2}{k}\ln\frac{x_M}{x_m},$$

which is at most a constant times $-\ln \epsilon$. Similarly, when $y < k/2a$ we have $\frac{dx}{dt} \ge kx/2$, so that x increases exponentially. Thus again the time spent on this segment is bounded by a constant times $-\ln \epsilon$. Finally, for $k/2a \le y \le 3k/2a$, observe that y is bounded by a constant, so that

$$bx - \ell \ln x = -k \ln \epsilon + O(1).$$

There are two connected components, one with $x < 1$ satisfying $-\ell \ln x = -k \ln \epsilon + O(1)$ and a second with $x > 1$, satisfying $bx > -k/2 \ln \epsilon$. On the first component,

$$\frac{dy}{dt} = \ell y - bxy \ge \frac{\ell}{2}y \ge \frac{\ell k}{4a},$$

at least when ϵ is sufficiently small, since throughout the curve x is small (tending to 0) as ϵ tends to 0. Thus the time spent on the first component is bounded by

$$\frac{3k/2a - k/2a}{\ell k/4a} \le \frac{4}{\ell},$$

a constant. Similarly, on the second component, x tends to ∞ as $\epsilon \to 0$, so if ϵ is sufficiently small,

$$\frac{dy}{dt} = \ell y - bxy \le \frac{-\ell}{2}y \le \frac{-\ell k}{4a}.$$

Again, the time spent on this segment is bounded by a constant. This completes the upper bound.