Math 53H: Take-Home Midterm Exam

Due in class on Wednesday, May 2, 2012

1. Consider the equation

$$\dot{x} = x + te^t.$$

Find a coordinate change $u = u(x,t),\ v = v(x,t)$ in the extended phase space such that in new coordinates the equation takes the form $\frac{du}{dv} = 0$.

2. Consider the equation

$$y' = f\left(\frac{y}{x}\right),\ y, x > 0,$$

where f is a smooth function on $\mathbb{R}$ such that

- there exists a unique value k such that $f(k) = k$;
- $k > 0$ and $f'(k) < 1$.

Find all solutions $y = \phi(x)$ of this equation which satisfy the condition $\lim_{x\to 0} \frac{\phi(x)}{x} = k$.

3. Consider the differential equation

$$\dot{x} = \mu t + \sin x.$$

Let $\phi_\mu(t)$ be the solution of this equation which satisfies the initial condition $\phi_\mu(0) = 2\mu$. Compute

$$\frac{\partial \phi_\mu}{\partial \mu}(t)\Big|_{\mu=0}.$$

4. Consider the equation

$$y' \sin 2x = 2(y + \cos x), x \in (0, \frac{\pi}{2}).$$

Find all solutions of this equation which remain bounded when $x \to \frac{\pi}{2}$.

5. In $\mathbb{R}^{2n}$ with coordinates $x_1, y_1, \ldots, x_n, y_n$ denote

$$\omega := \sum_1^n dx_i \wedge dy_i, \quad \Omega := dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Suppose that a vector field X on $\mathbb{R}^{2n}$ satisfies the equation $d(X \lrcorner \omega) = \omega$. Find $t \in \mathbb{R}$ such that

$$(X^t)^* \Omega = \frac{\Omega}{2}.$$

Each problem is 10 points.

You are allowed to consult the textbook and notes, but please do not discuss the work with classmates or anybody else.