Math 53H: Homework N8

Due to Friday, June 1

1. Consider a system

$$\dot{x} = f(t, x), \quad x \in \mathbb{R}^2$$

with a periodic right-hand side: $f(t+T, x) = f(t, x)$, $t \in \mathbb{R}, x \in \mathbb{R}^2$.

Suppose that $f_t(0) = 0$ for all $t \in \mathbb{R}$ and 0 is a Lyapunov stable equilibrium point of the system. Suppose also that the phase flow of this system preserves the area form on $\mathbb{R}^2$. Prove that 0 remains Lyapunov stable equilibrium point after linearization. Give an example which shows that this is wrong for asymptotic stability.

2. Consider the system

$$\dot{x} = y$$
$$\dot{y} = -x^2$$

and

$$\dot{x} = x^2 y$$
$$\dot{y} = -xy^2 \,.$$

Find all equilibrium points and study their stability (in Lyapunov and asymptotic senses)

3. The system of second order differential equations

$$\begin{aligned} m\ddot{x}_1 &= -k(x_1 - x_4) - k(x_1 - x_2) \\ m\ddot{x}_2 &= -k(x_2 - x_1) - k(x_2 - x_3) \\ m\ddot{x}_3 &= -k(x_3 - x_2) - k(x_3 - x_4) \\ m\ddot{x}_4 &= -k(x_4 - x_3) - k(x_4 - x_1), \end{aligned}$$

$m, k > 0$, describes small oscillations of 4 identical masses m positioned at the vertices of a square which are cyclicly connected by 4 identical elastic springs and can oscillate in the direction perpendicular to the plane of the square. (We may assume that the springs are stretched, but the masses are confined on vertical rods and can only slide along them without friction.)

Find the principal frequencies and the corresponding principal oscillations. Describe the mechanical meaning of the principal oscillations.

4. Consider a system of Newton equations

$$\ddot{x} = -\nabla U(x),\ x = (x_1, x_2) \in \mathbb{R}^2.$$

where the potential energy function U has the form $U(x_1, x_2) = u(x_1 - x_2)$ for a function $u : \mathbb{R} \to \mathbb{R}$. Find a first integral of this system, different from the energy integral.

5. Consider a differential equation

$$y' = f(x, y),$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function. Let $\phi_t(x)$ be the solution of this equation which satisfies the initial condition $\phi_t(0) = t$. Prove that

$$\frac{\partial \phi_t(x)}{\partial t} > 0$$

for all $t, x \in \mathbb{R}$.

Each problem is 10 points.