Math 53H: Solution to Problem 1 in Homework N8

1. Consider a system

$$\dot{x} = f(t, x), \quad x \in \mathbb{R}^2$$

with a periodic right-hand side: $f(t+T, x) = f(t, x)$, $t \in \mathbb{R}, x \in \mathbb{R}^2$. Suppose that $f_t(0) = 0$ for all $t \in \mathbb{R}$ and 0 is a Lyapunov stable equilibrium point of the system. Suppose also that the phase flow of this system preserves the area form on $\mathbb{R}^2$. Prove that 0 remains Lyapunov stable equilibrium point after linearization. Give an example which shows that this is wrong for asymptotic stability.

First let me point out that the last sentence of the problem does not make much sense. Indeed, *area preserving map can never be asymptotically stable.* Second, the claim of this exercise is wrong. There is counter-example to the claim when the matrix of the period map for the linearized flow is equal $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The correct statememt which we prove below is the following

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Let us denote by P the period map of the flow of the vector field f, and by A the period map of the linearized flow (which we will not distinguish from its matrix). Then $P(0) = 0$ and $A = d_0P$. The condition that P preserves the area form means that the determinant of the differential d_xP at every point x is equal to 1, and in particular, $\det A = 1$. Hence, Lyapunov stability of the linearized flow means that $|\mathrm{Tr}A| \leq 2$ and if $|\mathrm{Tr}A| = 2$ then $A = \pm I$. If $\mathrm{Tr}A = 2$ then both eihenvalues are either equal to 1 or to -1. Hence, if in that case $A \neq \pm I$ then the matrix A is not diagonalizable which contradicts the assumption.

Hence, we only need to rule out the case $|\mathrm{Tr}A| > 2$. Suppose that this is the case. Then A has real eigenvalues λ_1, λ_2 of the same sign such that $|\lambda_1| > 1 > |\lambda_2| > 0$. In this case the matrix diagonalizable, and hence we can choose coordinates when it is diagonal. Then the map P has the form

$$P(x, y) = (X(x, y), Y(x, y)) = (\lambda_1 x + \alpha(x, y), \lambda_2 y + \beta(x, y)),$$

where

$$\alpha(x, y), \beta(x, y) = o(|x| + |y|).$$

Choose $\epsilon > 0$ such that

$$\epsilon < \min\left(\frac{|\lambda_1| - 1}{3}, 1 - |\lambda_2|\right).$$

Denote $c := |\lambda_1| - 3\epsilon, b := |\lambda_2| + \epsilon$. Then $0 < b < 1 < c$.

There exists a neighborhood U of the origin such that

$$|\alpha(x, y)|, |\beta(x, y)| < \epsilon(|x| + |y|).$$

Lemma. *Suppose that $(x, y) \in U$ and $|y| \leq |x|$. Then*

$$|Y(x, y)| \leq |X(x, y)| \quad \textit{and} \quad |X(x, y)| \geq a|x|.$$

Proof. We have

$$|Y(x,y)| \leq |\lambda_2||y| + \epsilon(|x|+|y|) \leq b|y| + \epsilon|x|$$

and

$$\begin{aligned}|X(x,y)| \geq & |\lambda_1||x| - \epsilon(|x|+|y|) \geq (\lambda_1| - 3\epsilon)|x| + 2\epsilon|x| - \epsilon|y| \geq \\ & c|x| + \epsilon|x|.\end{aligned}$$

Hence,

$$|X(x,y)| \geq c|x| + \epsilon|x| \geq b|y| + \epsilon|x| \geq |Y(x,y)|.$$

and

$$|X(x,y)| \geq c|x| \geq ||x||.$$

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According to the definition of Lyapunov stability, if we take any point $(x_0, y_0) \in U$ sufficiently close to 0 and begin iterating the map P:

$$(x_{k+1}, y_{k+1}) = P(x_k, y_k), \quad k = 0, 1 \ldots,$$

then the sequence of points (x_k, y_k) should forever remain in U. On the other hand, if we choose $(x_0, y_0) \neq (0,0)$ with $|x_0| \geq |y_0|$ then according to Lemma we have $|x_k| \geq |y_k|$ for all $k \geq 0$ and hence, $|x_k| \geq c^k|x_0| \underset{k\to\infty}{\to} \infty$, which is a contradiction.