Math 53H: Homework N3

Due to Friday, April 27

1. Find the solution of the quasi-homogeneous equation

$$t\dot{x} - x = t \tan \frac{x}{t}, \quad t > 0$$

which satisfies the initial condition $x = \frac{\pi}{6}, t = 1$.

2. Consider an equation

$$\dot{x} + x = f(t), \tag{1}$$

where $f : \mathbb{R} \to [-M, M]$ is a bounded function. Prove that equation (1) has the unique solution bounded for all $t \in \mathbb{R}$ and find this solution. Show that this solution is periodic if the function f is periodic.

3. Show that the equation

$$\dot{x} = 2x \cos^2 t - \sin t$$

has the unique periodic solution and find this solution explicitly.

4. Let $\phi_u(t)$ be the solution of the equation

$$\dot{x} = x + x^2 + tx^3,$$

which satisfies the initial condition $\phi_u(2) = u$. Compute $\frac{\partial \phi_u}{\partial u}(t)|_{u=0}$.

5. Consider the plane field ξ in $\mathbb{R}^3$ given by a Pfaffian equation $dz - ydx = 0$.

a) Show that the flow of the vector field $Z = 2\frac{\partial}{\partial z} + 2y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$ preserves the plane field ξ, i.e. $Z^t_*\xi = \xi$ for all $t \in \mathbb{R}$.

b)* *Extra-credit.* Prove that the plane field ξ cannot be rectified even locally, i.e. there is no diffeomorphism $f : U \to W$ between any two neighborhoods U, W such that

$$f_*\xi = \{dz = 0\}.$$

Each (sub)problem is 10 points.