Math 53H: Final Exam

June 8, 2012

1. Consider the equation system

$$\ddot{x} + x = 2\mu \sin t + \mu x^2 \tag{1}$$

Let $\phi(t,\mu)$ be the solution of this system with the initial data $x(0) = 0, \dot{x}(0) = 0$. Find $\frac{\partial \phi}{\partial \mu}(t,0)$.

We have $\phi(t,0) = 0$, and thus, linearizing equation along this solution we get

$$\ddot{y} + y = 2\sin t. \tag{2}$$

We need to solve this equation with the initial data $y(0) = \dot{y}(0) = 0$. We use the method of complex amplitudes and consider the equation

$$\ddot{y} + y = 2e^{it}.$$

We will find a particular solution of this equation and then will take the imaginary part.

The characteristic polynomial is λ^2+1 and its roots $\pm i$. So this is the resonance case, and a particular solution of the inhomogeneous equation should be sought in the form $y = cte^{it}$. Plugging it to equation (2) we get $2cie^{it} = 2e^{it}$, or $c = -i$. Thus $y = -ite^{it}$. Hence, $\operatorname{Im} y = -t\cos t$ is a particular solution of (2). The general solution of equation (2) is equal to $-t\cos t + a\sin t + b\cos t$. Plugging the initial data we get $b = 0, a = 1$. Hence the required solution (2) which is equal to the derivative $\frac{\partial \phi}{\partial \mu}(t, 0)$ is equal to

$$(-t\cos t + \sin t).$$

2. Solve the system

$$\dot{x} = Atx + x, \ \ x \in \mathbb{R}^2,$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Find the fundamental system of solutions of this syste

For any 2 values of t the matrices $(At + I)$ commute, and hence the fundamental system of solutions of this system is formed by the columns of the matrix

$$e^{\int_0^t (As+I)ds} = e^{\frac{At^2}{2} + tI} = e^t e^{\frac{At^2}{2}}.$$

Let (x, y) be the coordinate system in $\mathbb{R}^2$. Introducing complex coordinate $z = x + iy$ the operator of multiplication by the matrix A in the real plane is the same is the operator of multiplication by i in $\mathbb{C}$.

Hence, $e^{\frac{At^2}{2}}$ in complex notation is equal to the complex number $e^{i\frac{t^2}{2}} = \cos\frac{t^2}{2} + i\sin\frac{t^2}{2}$. Returning to the real notation we get

$$e^{\frac{At^2}{2}} = \begin{pmatrix} \cos\frac{t^2}{2} & -\sin\frac{t^2}{2} \\ \sin\frac{t^2}{2} & \cos\frac{t^2}{2} \end{pmatrix}.$$

Hence,

$$e^t e^{\frac{At^2}{2}} = \begin{pmatrix} e^t\cos\frac{t^2}{2} & -e^t\sin\frac{t^2}{2} \\ e^t\sin\frac{t^2}{2} & e^t\cos\frac{t^2}{2} \end{pmatrix}.$$

The columns of this matrix is a fundamental system of solutions.

3.Find all equilibrium points of the system

$$\dot{x} = y$$

$$\dot{y} = \sin(x + y).$$

Study their Lyapunov and asymptotic stability. Sketch the phase curves of this system.

The equilibrium points are $A_k = (k\pi, 0)$. If k is even then the linearized system at A_k is

$$\dot{x} = y$$
$$\dot{y} = x + y.$$

If k is odd then the linearized system at A_k is

$$\dot{x} = y$$
$$\dot{y} = -x - y.$$

Suppose first that k is even. Then the characteristic polynomial is $\lambda^2 - \lambda - 1$ and the eigenvalues are real $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. One of them has a positive real part, and hence the point A_k is Lyapunov unstable.

If k is odd then the characteristic polynomial is $\lambda^2 + \lambda + 1$ and the eigenvalues are complex with negative real part: $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Hence the point A_k is asymptotically stable.

4. Consider the system

$$\dot{x} = -y^3$$
$$\dot{y} = x^3.$$

a) Compute the area of $f_{2012}(D)$, if D is the unit disc $D = \{x^2 + y^2 \leq 1\}$.

b) *Extra-credit.* Prove that the flow f_t is defined for all $t \in \mathbb{R}$.

a) The divergence of the vector field

$$X = -y^3 \frac{\partial}{\partial x} + x^3 \frac{\partial}{\partial y}$$

is equal to 0. Hence the phase flow preserves the area.

b) The system is Hamiltonian and has the integral $H = x^4 + y^4$. Hence, the flow preserves the level sets of this function which are compact. Hence, the flow is definied for all time.

5. It is known that the first coordinate of the one of the solutions of a real linear system with constant coefficients

$$\dot{x} = Ax, \ x \in \mathbb{R}^3,$$

is equal to $\cos t + e^{-t}$. Prove that the origin is a Lyapunov stable equilibrium point for this system.

The form of the solution implies that the eigenvalues of A are $-1, \pm i$. Hence, in an appropriate basis the matrix of the phase flow is equal to

$$C(t) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix},$$

and therefore for an initial data in a ball of any radius ϵ (with respect to the dot-product is this coordinate system) the solution remains in this ball, i.e. the origin is Lyapunov stable.

6. *Extra-credit.* Let X be a smooth vector field on $\mathbb{R}^2$ such that its phase flow X^t exists for all $t \in \mathbb{R}$. Show that $X^1 \neq F$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map with the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

The matrix has a negative determinant, and hence the map is orientation reversing, but on the other hand, the phase flow X^t of any system consists of orientation preserving diffeomorphisms, because $X^0 = \mathrm{Id}$, and hence by continuity the determinant of the Jacobi matrix is positive everywhere for all t.