Math 53H: Preparation problems for the Final Exam

1. Consider a system

$$\dot{x}_1 = x_2 - x_1 + x_1^2$$

$$\dot{x}_2 = x_2 + 3x_1 + x_1 x_2$$

Let $\phi(t, \alpha) = (\phi_1(t, \alpha), \phi_2(t, \alpha))$ be the solution of this sytem with the initial data $x_1(0) = \alpha, x_2(0) = 0$. Find $\frac{\partial \phi}{\partial \alpha}(t, 0)$.

2. Consider a system

$$\dot{x} = F(t)Ax,\ x \in \mathbb{R}^2,$$

where $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and $F(t) = 3t^2 + 2t + 1$. Find a fundamental system of solutions of this system.

3. Consider an equation

$$\ddot{x} + \alpha(t)\dot{x} + \omega^2 x = 0,$$

where $\alpha(t)$ and $\omega(t)$ are periodic functions with the period 1. Prove that if $\alpha(t) < 0$ for all $t \in \mathbb{R}$, then the equilibrium solution $x \equiv 0$ is not asymptotically stable.

4. Consider a system

$$\dot{x} = a(t)x + b(t)y$$

$$\dot{y} = -b(t)x - a(t)y,$$

where $a(t)$ and $b(t)$ are 2π-periodic functions such that

$$a(t) = \begin{cases} \lambda \sin t, & t \in [0, \pi], \\ 0, & t \in [\pi, 2\pi]; \end{cases}$$

$$b(t) = \begin{cases} 0, & t \in [0, \pi], \\ \frac{\pi}{6} \sin t, & t \in [\pi, 2\pi]; \end{cases}$$

Show that there exists a value $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ the equilibrium point $x = y = 0$ is Lyapunov stable, and for $\lambda > \lambda_0$ it is Lyapunov unstable.

5. Using the method of variation of constants find the general solution of the following system of differential equations:

$$\dot{x}_1 - x_2 + e^{-t} = 0$$
$$\dot{x}_2 - x_1 - e^{-t} = 0\,.$$

6. Find characteristic frequencies and characteristic oscillations of a system of two identical pendulums connected by a spring. This system is described by the Newton equations:

$$\ddot{q}_1 = -\frac{\partial U}{\partial q_1}$$
$$\ddot{q}_2 = -\frac{\partial U}{\partial q_2},$$

where

$$U = \frac{q_1^2}{2} + \frac{q_2^2}{2} + \frac{\alpha}{2}(q_1 - q_2)^2,\ \alpha > 0.$$

7. Consider the system

$$\dot{x} = ax + y$$
$$\dot{y} = ay - (2a + 1)x$$

Determine for each value of the parameter a whether the origin is an asymptotically or Lyapunov stable equilibrium point. Sketch the phase trajectories for $a = 1$.