Math 53H: Preparation problems for the Final Exam

1. Consider a system

$$\dot{x}_1 = x_2 - x_1 + x_1^2$$

$$\dot{x}_2 = x_2 + 3x_1 + x_1 x_2$$

Let $\phi(t,\alpha) = (\phi_1(t,\alpha), \phi_2(t,\alpha))$ be the solution of this sytem with the initial data $x_1(0) = \alpha, x_2(0) = 0$. Find $\frac{\partial \phi}{\partial \alpha}(t,0)$.

When $\alpha = 0$ the solution is easy to guess: $\phi(t,0) = (0,0)$. Consider the Taylor expansion of $\phi(t,\alpha)$ with respect to α at $(0,0)$ (we view t as a parameter):

$$\phi(t,\alpha) = \alpha A(t,\alpha) + o(\alpha).$$

Here $A = (a_1, a_2)$ is a vector-function.

Plugging this into the system and retaining only linear terms we get

$$\alpha \dot{a}_1 = \alpha a_2 - \alpha a_1$$

$$\alpha \dot{a}_2 = \alpha a_2 + 3\alpha a_1.$$

Equating coefficients with α we get

$$\dot{a}_1 = a_2 - a_1$$

$$\dot{a}_2 = a_2 + 3a_1$$

The characteristic polynomial of the matrix $\begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}$ is equal to $\lambda^2 - 4$. Hence, eigenvalues are ± 2. The eigenvector $Y^1 = (y_1, y_2)$ for the eigenvalue 2 can be found from the equation $-3y_1 + y_2 = 0$, i.e. $Y^1 = (1, 3)$. Similarly, the eigenvector Y^2 for the eigenvalue -2 is equal to $(-1, 1)$. Thus the general solution is

$$\begin{pmatrix} c_1 e^{2t} - c_2 e^{-2t} \\ 3c_1 e^{2t} + c_2 e^{-2t} \end{pmatrix}.$$

To find a solution which satisfies the initial data $(1, 0)$ we need to solve the system

$$c_1 - c_2 = 1$$

$$3c_1 + c_2 = 0;$$

. We compute $c_1 = \frac{1}{4}, c_2 = -\frac{3}{4}$, and thus

$$A = \begin{pmatrix} \frac{e^{2t} + 3e^{-2t}}{4} \\ \frac{3e^{2t} - 3e^{-2t}}{4} \end{pmatrix}.$$

2. Consider a system

$$\dot{x} = F(t)Ax, \ x \in \mathbb{R}^2,$$

where $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and $F(t) = 3t^2 + 2t + 1$. Find a fundamental system of solutions of this system.

The solution of this system which satisfies the initial condition $x(0) = x_0$ is

$$x(t) = e^{\left(\int\limits_0^t F(s)ds\right)A} x_0 = e^{(t^3 + t^2 + t)A} x_0.$$

Denote $u := t^3 + t^2 + t$. Note that the matrix A is in the Jordan normal form. Hence

$$e^{uA} = \begin{pmatrix} e^{2u} & ue^{2u} \\ 0 & e^{2u} \end{pmatrix}.$$

The fundamental system of solutions is given by columns of this matrix:

$$X_1 = \begin{pmatrix} e^{2u} \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} ue^{2u} \\ e^{2u} \end{pmatrix},$$

where $u = t^3 + t^2 + t$.

3. Consider an equation

$$\ddot{x} + \alpha(t)\dot{x} + \omega^2 x = 0,$$

where $\alpha(t)$ and $\omega(t)$ are periodic functions with the period 1. Prove that if $\alpha(t) < 0$ for all $t \in \mathbb{R}$, then the equilibrium solution $x \equiv 0$ is not asymptotically stable.

When converted to a 2-dimensional first order system $\dot{x} = y, \dot{y} = -\omega^2 x - \alpha(t)y$, it has the matrix $A(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\alpha(t) \end{pmatrix}$, and hence $\mathrm{Tr}A(t) = -\alpha(t) > 0$. The Wronskian $W(t)$ satisfies the equation $\dot{W}(t) = -\alpha(t)W(t)$. Hence

$$W(t) = W(0)e^{-\int\limits_0^t \alpha(s)ds} \underset{t\to\infty}{\longrightarrow} \infty.$$

Recall that the geometric meaning of the Wronskian is the coefficient of scaling the volume of a domain under the flow. If the system is asymptotically stable, then the volume of a sufficiently small domain should go to 0, and not to ∞ as in this case.

4. Consider a system

$$\begin{aligned} \dot{x} &= a(t)x + b(t)y \\ \dot{y} &= -b(t)x - a(t)y, \end{aligned} \tag{1}$$

where $a(t)$ and $b(t)$ are 2π-periodic functions such that

$$a(t) = \begin{cases} \lambda \sin t, & t \in [0, \pi], \\ 0, & t \in [\pi, 2\pi]; \end{cases}$$

$$b(t) = \begin{cases} 0, & t \in [0, \pi], \\ \frac{\pi}{6} \sin t, & t \in [\pi, 2\pi]; \end{cases}$$

Show that there exists a value $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ the equilibrium point $x = y = 0$ is Lyapunov stable, and for $\lambda > \lambda_0$ it is Lyapunov unstable.

The trace of the matrix of this system is equal to 0. Hence, the flow is area-preserving. and to determine whether the 0-solution is stable we need to compute the trace of the period map and analyze when it is < 2.

The period is equal to 2π. We compute the matrix A of the period map as the product $A = A_2 A_1$, where A_1 is the monodromy matrix of the map on the interval $[0, \pi]$ and A_2 is the monodromy matrix for the interval $[\pi, 2\pi]$.

When $t \in [0, \pi]$ the system (1) takes the form

$$\begin{aligned} \dot{x} &= \lambda(\sin t)x \\ \dot{y} &= -\lambda(\sin t)y, \end{aligned} \tag{2}$$

Hence, the solution is given by

$$x = x_0 e^{\lambda \int_0^t \sin s ds} = x_0 e^{\lambda(1-\cos t)},$$

$$y = y_0 e^{-\lambda \int_0^t \sin s ds} = y_0 e^{\lambda(\cos t - 1)}.$$

To compute the columns of the monodromy matrix A_1 we need to find solutions with the initial data $(1, 0)$ and $(0, 1)$ and plug $t = \pi$. Thus we get

$$A_1 = \begin{pmatrix} e^{2\lambda} & 0 \\ 0 & e^{-2\lambda} \end{pmatrix}.$$

When $t \in [\pi, 2\pi]$ the system (1) takes the form

$$\begin{aligned} \dot{x} &= \frac{\pi}{6}(\sin t)y \\ \dot{y} &= -\frac{\pi}{6}(\sin t)x, \end{aligned} \tag{3}$$

The solution $z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of this system on the interval $[\pi, 2\pi]$ can be written as

$$z(t) = e^{\frac{\pi}{6}\int\limits_{\pi}^{s} \sin s ds J} z(\pi) = e^{\frac{\pi}{6}(-1-\cos s)J} z(\pi),$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The matrix A_2 is equal to

$$e^{-\frac{\pi}{3}J} = e^{\begin{pmatrix} 0 & -\frac{\pi}{3} \\ \frac{\pi}{3} & 0 \end{pmatrix}}$$

It is easy to exponentiate this matrix observing that the matrix in the exponent is the matrix of multiplication by $\frac{\pi i}{3}$ on the plane $\mathbb{R}^2$ viewed as $\mathbb{C}$. Hence, $e^{-\frac{\pi}{3}J}$ is the matrix of multiplication by $e^{\frac{\pi i}{3}}$, i.e. the matrix of rotation by $\frac{\pi}{3}$. Thus,

$$A_2 = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

and

$$A = A_2 A_1 = \begin{pmatrix} \frac{e^{2\lambda}}{2} & -\frac{\sqrt{3}e^{2\lambda}}{2} \\ \frac{\sqrt{3}e^{-2\lambda}}{2} & \frac{e^{-2\lambda}}{2} \end{pmatrix}.$$

Denote $\mu := e^{2\lambda}$. Then $\mathrm{Tr}A = \frac{1}{2}\left(\mu + \frac{1}{\mu}\right)$. Hence the inequality $\mathrm{Tr}A > 2$ is equivalent to

$$\mu^2 - 4\mu + 1 > 0.$$

The roots of this quadratic polynomial are $\mu_\pm = 2 \pm \sqrt{3}$. we have $\mu_- = 2 - \sqrt{3} < 1, \mu_+ = 2 + \sqrt{3} > 1$. Respectively $\lambda_+ = \frac{1}{2}\ln(2+\sqrt{3}) > 0, \lambda_- = \frac{1}{2}\ln(2-\sqrt{3}) < 0$. Thus, $\lambda_0 = \lambda_+ = \frac{1}{2}\ln(2+\sqrt{3})$ is the required value of the parameter λ which separates the domains of Lyapunov stability and instability.

5. Using the method of variation of constants find the general solution of the following system of differential equations:

$$\dot{x}_1 - x_2 + e^{-t} = 0$$
$$\dot{x}_2 - x_1 - e^{-t} = 0\,.$$

Let us first solve the homogeneous system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1\,.$$

The vector functions $\phi_1(t) = (e^t, e^t)$ and $\phi_2(t) = (e^{-t}, -e^{-t})$ form the fundamental system of solutions of this system. According to the method of variations of constants, the solution $\psi(t)$ of the inhomogeneous system should be sought in the form

$$\psi(t) = c_1(t)\phi_1(t) + c_2(t)\phi_2(t).$$

Plugging into the system we get

$$\dot{c}_1\phi_1 + \dot{c}_2\phi_2 + (c_1\dot{\phi}_1 + c_2\dot{\phi}_2) = A(c_1\phi_1 + c_2\phi_2) = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix},$$

where we denoted by A the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Taking into account that ϕ_1 and ϕ_2 are solutions of the homogeneous system, we get

$$\dot{c}_1\phi_1 + \dot{c}_2\phi_2 = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix},$$

i.e.

$$\dot{c}_1 e^t + \dot{c}_2 e^{-t} = -e^{-t}$$
$$\dot{c}_1 e^t - \dot{c}_2 e^{-t} = e^{-t}\,,$$

$$\dot{c}_1 e^{2t} + \dot{c}_2 = -1$$
$$\dot{c}_1 e^{2t} - \dot{c}_2 = 1\,,$$

and hence $\dot{c}_1 = 0, \dot{c}_2 = -1$.

Thus $c_1 = a, c_2 = b - t$, where a, b are arbitrary constants. Thus the general solution of the inhomogeneous systemhas the form

$$\psi(t) = \begin{pmatrix} ae^t + (b-t)e^{-t} \\ ae^t + (t-b)e^{-t} \end{pmatrix}.$$

6. Find principal frequencies and principal oscillations of a system of two identical pendulums connected by a spring. This system is described by the Newton equations:

$$\ddot{q}_1 = -\frac{\partial U}{\partial q_1}$$
$$\ddot{q}_2 = -\frac{\partial U}{\partial q_2},$$

where

$$U = \frac{q_1^2}{2} + \frac{q_2^2}{2} + \frac{\alpha}{2}(q_1 - q_2)^2,\ \alpha > 0.$$

The quadratic form U can be written as

$$U = \frac{1}{2}\langle Aq, q\rangle,\ \ q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

where

$$A = \begin{pmatrix} 1+\alpha & -1 \\ -1 & 1+\alpha \end{pmatrix}.$$

The characteristic polynomial $\chi_A(\lambda)$ of the matrix A has the form

$$\chi_A(\lambda) = (\alpha - \lambda)^2 + 2(\alpha - \lambda) = (\alpha - \lambda)(\alpha + 2 - \lambda),$$

and hence the eigenvalues are $\lambda_1 = \alpha, \lambda_2 = \alpha + 2$. Corresponding basis of eigenvectors consists of $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This basis is orthogonal (but not orthonormal). The principal (characteristic) frequences are $\omega_1 = \sqrt{\alpha}$ and $\omega_2 = \sqrt{2 + \alpha}$. and the principal oscillations are

$$x_1 = A_1 \cos(\sqrt{\alpha} t + \theta_1), x_2 = A_1 \cos(\sqrt{\alpha} t + \theta_1)$$

and

$$x_1 = A_2 \cos(\sqrt{\alpha + 2} t + \theta_2), x_2 = -A_2 \cos(\sqrt{\alpha + 2} t + \theta_2).$$

7. Consider the system

$$\dot{x} = ax + y$$
$$\dot{y} = ay - (2a + 1)x$$

Determine for each value of the parameter a whether the origin is an asymptotically or Lyapunov stable equilibrium point. Sketch the phase trajectories for $a = 1$.

The matrix of the system is equal to

$$A = \begin{pmatrix} a & 1 \\ -(2a + 1) & a \end{pmatrix}.$$

The characteristic polynomial is equal to

$$\chi_A(\lambda) = \lambda^2 - 2a\lambda + (a + 1)^2.$$

The eigenvalues are $\lambda_{1,2} = a \pm \sqrt{-(2a + 1)}$.

If $a > -\frac{1}{2}$ the eigenvalues have non-zero imaginary part. The real part is negative when $a < 0$ and positive when $a > 0$. Hence, the system is asymptotically stable, when $a \in (-\frac{1}{2}, 0)$ and Lyapunov unstable when $a > 0$.

When $a = 0$ the system takes the form

$$\dot{x} = y$$

$$\dot{y} = -x.$$

All trajectories of this system are closed and hence it is Lyapunov stable, but is not asymptotically stable.

When $a \le -\frac{1}{2}$ the eigenvalues are real; one of them is always negative, and the other one is also negative unless $a = -1$. when it is $= 0$. In the latter case the system is Lyapunov but not asymptotically stable. In other cases it is asymptotically stable.

Summarizing, we get the the system is Lyapunov unstable for $a > 0$, Lyapunov stable, but asymptotically unstable for $a = 0$ and $a = -1$ and asymptotically stable for other a.