Math 53H Midterm Solutions

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1.) The homogeneous solution of this equation is $y(t) = ce^t$. Guessing a solution of form $x(t) = c(t)e^t$ leads to $c'(t) = t$, so a particular solution is given by $x(t) = \frac{t^2}{2}e^t$ and the general solution is $x(t) = (t^2/2 + c)e^t$. It is now evident that if we set $u(t) = e^{-t}x(t) - t^2/2$ then $\frac{du}{dt} = 0$. Indeed, we check that

$$\frac{du}{dt} = -e^{-t}x(t) + e^{-t}\dot{x}(t) - t = 0.$$

2.) The equation is homogeneous, so we make change of coordinates $(s,u) \mapsto (e^s, ue^s) = (x,y)$ with $u > 0$ and $s \in \mathbb{R}$. In these coordinates the equation takes the form

$$dy = f(y/x)dx \qquad \Leftrightarrow \qquad du = (f(u) - u)ds$$

and the limiting initial condition $\lim_{x \downarrow 0} \frac{y}{x} = k$ transforms into $\lim_{s \to -\infty} u(s) = k$.

Observe that the pair of conditions on f imply $f(u) > u$ for $u < k$ and $f(u) < u$ for $u > k$.

One solution of the differential equation satisfying the limit condition is $u \equiv k$. We are going to show that this is the only solution.

Suppose not, and suppose first that there exists $s_0 \in \mathbb{R}$ for which $u(s_0) = u_0 > k$. We claim that for all $s < s_0$, $u(s) \geq u_0$. Suppose not so that there exists $s_- < s_0$ with $u(s_-) < u_0$. Observe that $\frac{du}{ds}\big|_{s=s_0} = f(u_0) - u_0 < 0$ and therefore there exists a $\delta > 0$ such that for $s \in (s_0 - \delta, s_0)$, $u(s) > u_0$. Since $u(s_-) < u_0$ by the Intermediate Value Theorem there exists $s' \in (s_-, s_0 - \delta)$ with $u(s') = u_0$. Let s_1 be the supremum of all such $s' \in (s_-, s_0 - \delta)$ with $u(s') = u_0$. Then $u(s_1) = u_0$ by continuity. By the maximality of s_1, $u(s) > u_0$ for $s \in (s_1, s_0)$. Since $u(s_1) = u(s_0)$, by the Mean Value Theorem there exists $s_2 \in (s_1, s_0)$ with $u'(s_2) = 0$. But $u'(s_2) = f(u(s_2)) - u(s_2) < 0$ since $u(s_2) > u_0 > k$. This is a contradiction, so we deduce that $u(s_0) = u_0 > k$ implies $u(s) > u_0$ for all $s < s_0$, and therefore, if there exists an $s \in \mathbb{R}$ with $u(s) > k$, then it is not possible that $\lim_{s \to -\infty} u(s) = k$.

An almost identical argument shows that if there exists an s with $u(s) = u_0 < k$ then $\lim_{s \to -\infty} u(s) \leq u_0$. It follows that $\lim_{s \to -\infty} u(s) = k$ is only possible if $u \equiv k$, as wanted.

3.) The vector field of the differential equation is smooth. It follows that the solutions depend smoothly on the initial conditions. Therefore, we may differentiate the equation with respect to μ to find

$$\frac{\partial^2}{\partial t \partial \mu}\phi_\mu(t) = t + \cos(\phi_\mu(t))\frac{\partial}{\partial \mu}\phi_\mu(t).$$

Since $\phi_0(t) \equiv 0$ is a solution of the differential equation, which is unique by smoothness of the vector field, it follows that on evaluating the above equation at $\mu = 0$ we set $y = \frac{\partial}{\partial \mu}\phi_m u(t)\Big|_{\mu=0}$ to find

$$\dot{y} = t + y, \qquad y(0) = 2.$$

This equation has homogeneous solution $v(t) = ae^t$. Guessing a particular solution of form $c(t)e^t$, we find $c'(t) = te^{-t}$, or $c(t) = 1 - te^{-t} - e^{-t}$. Thus we arrive at the general solution

$$y(t) = ae^t - t - 1.$$

The initial condition implies $a = 3$, so we conclude

$$\frac{\partial \phi_\mu}{\partial \mu}(t)\Bigg|_{\mu=0} = 3e^t - t - 1.$$

4.) This is an inhomogeneous linear equation. Writing it in the form

$$\frac{dy}{dx} = y\frac{2}{\sin 2x} + \frac{2\cos x}{\sin 2x}$$

we see that the general solution is given by

$$y(x) = ce^{\int_{x_0}^{x} \frac{2du}{\sin 2u}} + \int_{x_0}^{x} e^{\int_{\xi}^{x} \frac{2du}{\sin 2u}} \frac{2\cos\xi}{\sin 2\xi} d\xi = e^{\int_{x_0}^{x} \frac{2du}{\sin 2u}} \left[c + \int_{x_0}^{x} e^{\int_{\xi}^{x_0} \frac{2dv}{\sin 2v}} \frac{2\cos\xi d\xi}{\sin 2\xi} \right].$$

Since $e^{\int_{x_0}^{x} \frac{2du}{\sin 2u}} \to \infty$ as $x \uparrow \frac{\pi}{2}$, it is only possible for the solution to remain bounded if

$$c = -\lim_{x\uparrow\frac{\pi}{2}} \int_{x_0}^{x} e^{\int_{\xi}^{x_0} \frac{2udu}{\sin 2u}} \frac{d\xi}{\sin\xi}$$

and so we obtain a unique candidate solution of

$$y = e^{\int_{x_0}^{x} \frac{2du}{\sin 2u}} \int_{\pi/2}^{x} e^{\int_{\xi}^{x_0} \frac{2du}{\sin 2u}} \frac{d\xi}{\sin\xi} = \int_{\pi/2}^{x} e^{\int_{\xi}^{x} \frac{2du}{\sin 2u}} \frac{d\xi}{\sin\xi}.$$

This can be simplified significantly. For $u \in (0, \pi)$, the antiderivative of $\frac{du}{\sin u}$ is $\ln \frac{\sin u}{1+\cos u}$. It follows that the above integral is equal to

$$\frac{\sin 2x}{1+\cos 2x} \int_{\pi/2}^{x} \frac{\sin\xi(1+\cos 2\xi)}{\sin 2\xi} d\xi = \tan x \int_{\pi/2}^{x} \cos\xi d\xi = \frac{\sin x(\sin x - 1)}{\cos x}.$$

This solution does indeed remain bounded throughout $(0, \pi/2)$, since $\cos x$ vanishes to order 1 at $\pi/2$, while $\sin x - 1$ vanishes to order 2.

5.) Recall the following facts regarding differential forms.

1. $\omega^{\wedge n} = n!\Omega$
2. Given vector X and k-form α, ℓ-form β, with $\alpha \wedge \beta \neq 0$,
$$X \rfloor (\alpha \wedge \beta) = (X \rfloor \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \rfloor \beta)$$
3. $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$.

From 2 it is straightforward to establish by induction that
$$X \rfloor \omega^{\wedge k} = k(X \rfloor \omega) \wedge \omega^{\wedge (k-1)}$$
for $k \leq n$. Thus
$$X \rfloor \Omega = \frac{1}{n!} X \rfloor \omega^{\wedge n} = \frac{1}{(n-1)!}(X \rfloor \omega) \wedge \omega^{\wedge (n-1)}.$$
Therefore, $d(X \rfloor \Omega) = \frac{1}{(n-1)!} d(X \rfloor \omega) \wedge \omega^{\wedge (n-1)} = \frac{1}{(n-1)!} \omega^{\wedge n} = n\Omega$.

Using Cartan's formula, we compute
$$\frac{d}{dt}(X^t)^* \Omega \bigg|_{t=0} = \mathcal{L}_X(\Omega) = X \rfloor d\Omega + d(X \rfloor \Omega) = n\Omega.$$
Since
$$\frac{d}{dt}(X^t)^* \Omega \bigg|_{t=t_0} = (X^{t_0})^* \mathcal{L}_X(\Omega) = n(X^{t_0})^* \Omega$$
we may set $(X^t)^*\Omega = f(t)\Omega$ to find $f'(t) = nf(t)$ and $f(0) = 1$, so $f(t) = e^{nt}$. It follows that $(X^t)^*\Omega = \frac{\Omega}{2}$ at $t = -\frac{\ln 2}{n}$.