Math 53H Homework 4 Solutions

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1.) As suggested in the hint we take $X = e^x \frac{d}{dx}$ and $Y = g(x)\frac{d}{dx}$ and attempt to solve $[X, Y] = X$. This will evidently suffice, since it will follow that each of the given Lie brackets is equal to $X \neq 0$.

Let ϕ be any smooth function. Then

$$[X, Y]\phi(x) = (XY - YX)\phi(x) = e^x \frac{d}{dx}(g(x)\phi'(x)) - g(x)\frac{d}{dx}(e^x \phi'(x)) = e^x(g'(x) - g(x))\phi'(x)$$

so $[X, Y] = e^x(g'(x) - g(x))\frac{d}{dx}$. It follows that solving $[X, Y] = X$ is equivalent to solving $g'(x) = g(x) + 1$. This is satisfied by $g(x) = e^x - 1$ and we conclude that the pair $X = e^x \frac{d}{dx}$, $Y = (e^x - 1)\frac{d}{dx}$ satisfy the given condition.

2.) We guess a linear solution $P_1 = Ap_1 + Bp_2$, $P_2 = Cp_1 + Dp_2$. Equating coefficients of $dp_i \wedge dq_j$ in

$$dP_1 \wedge dQ_1 + dP_2 \wedge dQ_2 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

leads to the equations

$$A + C = 1, \quad A - C = 0, \quad B + D = 0, \quad B - D = 1$$

and hence $A = B = C = 1/2$ and $D = -1/2$. Hence $P_1 = p_1/2 + p_2/2$, $P_2 = p_1/2 - p_2/2$.

3.) Let θ be the angle that the pendulum makes with the downward normal. After scaling, we may assume that the potential energy is given by $U(\theta) = -\cos\theta$. With this normalization, the critical energy is $E_1 = 1$, since this is the potential energy at the unstable equilibrium $\theta = \pm\pi$. We study the period of oscillation as $E \uparrow E_1 = 1$.

Given $E < 1$, the limit points of motion are given by $U(\theta) = -\cos\theta = E$, that is $a, b = \pm\cos^{-1}(-E)$. We make the convention that $\cos^{-1}(x) \in [0, \pi)$. According to p. 147 eqn (4), the period $T(E)$ is given by

$$T(E) = 2\int_a^b \frac{d\xi}{\sqrt{2(E - U(\xi))}} = 2\int_{-\cos^{-1}(-E)}^{\cos^{-1}(-E)} \frac{d\xi}{\sqrt{2(E + \cos\xi)}}.$$

By symmetry, this is

$$\frac{4}{\sqrt{2}}\int_0^{\cos^{-1}(-E)} \frac{d\xi}{\sqrt{E + \cos\xi}}. \tag{1}$$

Expand $\cos\xi$ in its Taylor series about $u = \cos^{-1}(-E)$ to find

$$\begin{aligned}\cos\xi &= -E - (\xi - u)\sin u - \frac{(\xi - u)^2}{2}\cos u + O((u - \xi)^3) \\ &= -E + (u - \xi)\sqrt{1 - E^2} + \frac{E}{2}(\xi - u)^2 + O((u - \xi)^3)\end{aligned}$$

and therefore

$$E + \cos\xi = (u - \xi)\sqrt{1 - E^2} + \frac{E}{2}(\xi - u)^2 + O((u - \xi)^3). \tag{2}$$

Here the $O(...)$ notation should be understood as indicating a quantity bounded by a fixed constant times $|u - \xi|^3$ for all $0 \le \xi \le u$. Moreover, the fixed constant may be taken independent of E since all derivatives of $\cos\theta$ are bounded.

Now set $1 - E = \delta$ and assume δ is very small, say $< 10^{-5}$ (so that $\log\log\frac{1}{\delta}$ is well defined). In the integral of (1), make change of variables $\zeta = u - \xi$ and split the integral to obtain

$$\int_0^u \frac{d\zeta}{\sqrt{E + \cos(u - \zeta)}} = \left\{\int_0^{\delta^{1/2}\log 1/\delta} + \int_{\delta^{1/2}\log 1/\delta}^{(\log(\log 1/\delta))^{-1}} + \int_{(\log(\log 1/\delta))^{-1}}^u\right\}\frac{d\zeta}{\sqrt{E + \cos(u - \zeta)}}.$$

Only the middle integral is going to make a substantial contribution. We first evaluate this integral. Since

$$\sqrt{1 - E^2} = \sqrt{(1 - E)(1 + E)} < \sqrt{2\delta},$$

for $\delta^{1/2}\log(1/\delta) < \zeta < (\log\log\delta)^{-1}$ we have $\zeta\sqrt{1 - E^2} < \sqrt{2\delta}\zeta = O(\frac{\zeta^2}{\log 1/\delta})$, while $\zeta^3 = O((\log\log\delta)^{-1}\zeta^2)$. [Here, the $O(EXP)$ should be taken to mean: bounded by a fixed constant times the expression EXP, for all δ sufficiently small and ζ in the stated range.]

Thus on the second integral,

$$E + \cos(u-\zeta) = (1+o(1))\frac{E}{2}\zeta^2 = (1+o(1))\frac{\zeta^2}{2},$$

and therefore the middle integral is

$$(1+o(1))\int_{\delta^{1/2}\log 1/\delta}^{(\log(\log 1/\delta))^{-1}} \frac{d\zeta}{\sqrt{\zeta^2/2}} = (\sqrt{2}+o(1))\left[\log((\log\log 1/\delta)^{-1}) - \log(\sqrt{\delta}) - \log\log\frac{1}{\delta}\right]$$
$$= -\frac{\sqrt{2}}{2}\log\delta + o(\log\delta).$$

[The notation $o(1)$ here means a quantity that becomes smaller in size than any fixed constant, as $\delta \to 0$. The relative notation $A = (1+o(1))EXP$ is really a pair of inequalities $(1-\epsilon)EXP < A < (1+\epsilon)EXP$; for any fixed $\epsilon > 0$ this pair becomes true once δ is sufficiently small. The various $'o'$ notations can be obtained by inserting these inequalities at each step.]

We now show that the first and third integrals are of smaller order in δ. The first integral we write as

$$\int_0^{\delta^{1/2}\log 1/\delta} \frac{d\zeta}{\sqrt{(1+o(1))(\sqrt{\delta}\zeta + \zeta^2/2) + O(\zeta^3)}}$$
$$\leq (1+o(1))\int_0^{\delta^{1/2}} \frac{d\zeta}{\sqrt{\delta^{1/2}\zeta}} + \int_{\delta^{1/2}}^{\delta^{1/2}\log 1/\delta} \frac{d\zeta}{\sqrt{\zeta^2/2}} = O(1) + O(\log\log 1/\delta)$$

We observe that the length of the third integral is $O(1)$, and that the integrand is decreasing in ζ. Therefore the net integral is bounded by the size of the integrand at the lower limit, which is $(1+o(1))\log\log 1/\delta$, since the first and third terms of the Taylor series are asymptotically smaller than the second.

Putting the three integrals together, we deduce that the period is

$$(1) = -2\log\delta + o(\log\delta)$$

as wanted.