Math 52H: Practice problems for the midterm

1. Let $V$ be an $n$ dimensional vector space and $A : V \to V$ a linear operator. Prove that the rank of the operator $A^* : \Lambda^{n-1}(V^*) \to \Lambda^{n-1}(V^*)$ can only take values 0, 1 and $n$.

Choose some coordinates $x_1, \ldots, x_n$ in $V$ and denote $l_i := A^* x_i$, $i = 1, \ldots, n$.

The forms $\alpha_i := x_1 \wedge \cdots \wedge \overset{i}{\vee} \wedge \cdots \wedge x_n$, $i = 1, \ldots, n$, form a basis of $\Lambda^{n-1}(V^*)$ and we have $A^* \alpha_i = l_1 \wedge \cdots \wedge \overset{i}{\vee} \wedge \cdots \wedge l_n$. If $A$ has rank $n$, then the linear forms $l_i$ are linearly independent, and hence the forms $A^* \alpha_i$, $i = 1, \ldots, n$, form a basis of $\Lambda^{k-1}(V^*)$. Therefore the rank of $A^* : \Lambda^{n-1}(V^*) \to \Lambda^{n-1}(V^*)$ in this case is equal to $n$. If rank$A < n$, then one of the forms $l_i$, say $l_n$, is a linear combination of the others: $l_n = \sum_{i=1}^{n-1} c_i l_i$, $c_i \in \mathbb{R}$. This implies that $A^* \alpha_i$ is proportional to $\alpha_n = l_1 \wedge \cdots \wedge l_{n-1}$ for each $i$, and hence the image of the map $A^* A$ has dimension 1 if $\alpha_n \neq 0$ (and this happens if rank$A = n - 1$). In this case then rank$A^* = 1$. Otherwise, it is 0.

2. Consider vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ and $v'_1 \in \mathbb{R}^n$. Prove that

$$\text{Vol}_n P(v_1 + v'_1, v_2, \ldots, v_n) \leq \text{Vol}_n P(v_1, v_2, \ldots, v_n) + \text{Vol}_n P(v'_1, v_2, \ldots, v_n)$$

and find out when the above inequality is an equality.

We have

$$\text{Vol}_n P(v_1 + v'_1, v_2, \ldots, v_n) = |x_1 \wedge \cdots \wedge x_n(v_1 + v'_1, v_2, \ldots, v_n)|$$

$$= |x_1 \wedge \cdots \wedge x_n(v_1, v_2, \ldots, v_n) + x_1 \wedge \cdots \wedge x_n(v'_1, v_2, \ldots, v_n)|$$
\[ \leq |x_1 \wedge \cdots \wedge x_n(v_1, v_2, \ldots, v_n)| + |x_1 \wedge \cdots \wedge x_n(v'_1, v_2, \ldots, v_n)| \]

\[ = \text{Vol}_n P(v_1, v_2, \ldots, v_n) + \text{Vol}_n P(v'_1, v_2, \ldots, v_n). \]

The equality holds if and only if when

- either one of the terms is 0, i.e. the corresponding vectors are linearly dependent, or
- if both terms have the same sign, i.e. when the bases \(v_1, \ldots, v_n\) and \(v'_1, \ldots, v_n\) define the same orientation of the ambient space.

3. Given a non-zero exterior \(k\)-form \(\alpha\) let us denote \(A(\alpha) = \{l \in V^*; \alpha \wedge l = 0\}\). Then \(A(\alpha)\) is a linear subspace of \(V^*\). Prove

a) \(\dim A(\alpha) \leq k\);

b) \(\dim A(\alpha) = k\) if and only if \(\alpha\) is decomposable, i.e. \(A(\alpha) = l_1 \wedge \cdots \wedge l_k\) for some 1-forms \(l_1, \ldots, l_k\).

Denote \(m := \dim A(\alpha)\). One can choose coordinates in \(V\) such that \(\text{Span}(x_1, \ldots, x_m) = A(\alpha)\). Let us write \(\alpha = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} x_{i_1} \wedge \cdots \wedge x_{i_k}\). By assumption, \(x_j \wedge \alpha = 0\) for any \(j \leq m\). It follows that each term entering \(\alpha\) should contain each of the variables \(x_1, \ldots, x_m\). Therefore \(m \leq k\). On the other hand, if \(m = k\) then \(\alpha = ax_1 \wedge \cdots \wedge x_k\).

4. Let \(A \subset \mathbb{R}\) be a connected set. Prove that \(A\) is either a point, or an interval (open, closed or semiopen), or a ray (open or closed), or the whole line \(\mathbb{R}\).

If \(A\) contains two points \(a < b\) then it contains the whole interval \([a, b]\). Indeed, if \(c \in [a, b]\) and \(c \notin A\). Then we can write \((A = (-\infty, c) \cap A) \cup (A \cap (c, \infty))\), i.e. we present \(A\) as the union of two relatively open and non-empty sets, i.e. \(A\) is disconnected which contradicts our assumption. Now denote \(m := \inf A, M := \sup A\), where we allow \(m = -\infty\) and/or \(M = \infty\). the previous argument then show that \(A\) is either \((m, M)\) or \([m, M]\), \((m, M]\), or \([m, M]\) depending on whether infimum and maximum belong to \(A\) or not.