1. Show that any bilinear function $\Phi$ can be presented in a unique way as a sum

$$\Phi = S + A,$$

where $S$ is a symmetric and $A$ is skew-symmetric bilinear functions. Show that a similar statement is not true for 3-linear functions on $\mathbb{R}^3$, and give an example of a 3-linear function which cannot be written as a sum of a symmetric and an anti-symmetric functions.

2. Recall that the trace $\text{Tr} A$ of a square matrix $A$ is the sum of its diagonal elements. Let $M_2$ denote the vector space of symmetric $2 \times 2$ matrices with real entries. Show that the formula

$$\langle X, Y \rangle = \text{Tr}(XY)$$

defines an inner product on $M_2$ and find the matrix of this bilinear form in the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. Let $f$ and $g$ be bilinear functions on $\mathbb{R}^n$ with matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$, respectively. Find the expression of the 4-tensor $f \otimes g$ in the basis

$$x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}, \quad 1 \leq i_1, i_2, i_3, i_4 \leq n.$$
4. Let $\alpha$ be an exterior 2-form, and $\beta$ is a 1-form on a 3-dimensional space. Suppose that $\alpha \wedge \beta = 0$. Prove that there exists a 1-form $\gamma$ such that $\alpha = \beta \wedge \gamma$.

5. Let \[ \theta = \sum_{i=1}^{n-1} x_i \wedge x_{i+1} \] be an exterior 2-form on $\mathbb{R}^n$, and $A, B \in \mathbb{R}^n$ are vectors

$$A = (1, 1, 1, \ldots, 1), \quad B = (-1, 1, -1, \ldots, (-1)^n).$$

Compute $\theta(A, B)$.

6. Denote coordinates in the space $\mathbb{R}^{2n}$ by $(x_1, x_2, \ldots, x_{2n-1}, x_{2n})$. Consider a 2-form

$$\omega = x_1 \wedge x_2 + x_3 \wedge x_4 + \ldots x_{2n-1} \wedge x_{2n}$$

and the map $C_\omega$ of the space $V = \mathbb{R}^{2n}$ to the dual space $V^*$ given by the formula

$$C_\omega(v)(Z) = \omega(v, Z), \quad v, Z \in V.$$ 

Find the matrix $C$ of the map $C_\omega$ with respect to the standard basis in $V = \mathbb{R}^{2n}$ and its dual basis in $V^*$.

Each problem is 10 points.