1. Let $SO(3)$ denote the group of orthogonal $3 \times 3$ matrices with determinant $+1$ and $UT(S^2)$ denote the unit tangent bundle to $S^2$, i.e. the space of pairs $(x, y)$ where $x \in \mathbb{R}^3$ is a vector of length $1$, $||x|| = 1$, and $y \in \mathbb{R}^3_x$ is another unit vector tangent to the sphere $S^2$ at the point $x$.

a) Introduce $C^\infty$-manifold structure on $SO(3)$ and $UT(S^2)$, i.e. describe coordinate charts and transition maps.

b) Prove that $SO(3)$ and $UT(S^2)$ are diffeomorphic.

c) Prove that the above manifolds are diffeomorphic to the real projective plane $\mathbb{R}P^3$.

2. The tangent bundle $T(S^n)$ of a unit $n$-dimensional sphere $S^n = \{ \sum_{i=1}^{n+1} x_i^2 = 1 \} \subset \mathbb{R}^{n+1}$ is defined as the set of pairs $(x, y)$, where $x \in S^n$ and $y \in \mathbb{R}^{n+1}_x$ is a tangent vector to $S^n$ at the point $x$. Describe the manifold structure on $T(S^n)$ and prove that $T(S^3)$ is diffeomorphic to $S^3 \times \mathbb{R}^3$.

Hint. Use the fact that the vectors in $\mathbb{R}^4$ can be viewed as quaternions. One can write a vector $(x, y, z, u)\mathbb{R}^4$ as a quaternion $x + yi + zj + uk$, and the multiplication table is given by $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. 

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3. Prove that there is no diffeomorphism \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) which maps the lines
\[
l_1 = \{ y = 0 \} , \quad l_2 = \{ x = y \} , \quad l_3 = \{ x = 0 \} , \quad l_4 = \{ x = -y \}
\]
onto the lines
\[
l_1 = \{ y = 0 \} , \quad \tilde{l}_2 = \{ x = 2y \} , \quad l_3 = \{ x = 0 \} , \quad l_4 = \{ x = -y \},
\]
respectively.

4. Prove that \( \mathbb{R}^+ = \{ x \geq 0 \} \subset \mathbb{R} \) is not homeomorphic to \( \mathbb{R} \).

5. The mean value of a function \( f \) over a domain \( U \subset \mathbb{R}^n \) is defined as
\[
\text{mean}(f) = \frac{\int_U f \, dV}{\text{Vol}(U)}.
\]
Compute the mean value of the function
\[
f(x, y, z) = e^{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2}}}
\]
over the solid ellipsoid
\[
E := \left\{ \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} \leq 1 \right\}.
\]
Each problem (and each subproblem in Problem 1) is 10 points.