Math 52H: Homework N6

Due to Friday, February 20

1. Compute the integral
   \[ \iint_D \left| \frac{x+y}{2} - x^2 - y^2 \right| \, dx \, dy, \]
where \( D \) is the unit disc \( \{ x^2 + y^2 \leq 1 \} \).

2. Compute the integral
   \[ \iint_{D_a} x^{20} y^{15} \, dx \, dy, \]
where \( D_a = \{ x^2 + y^2 \leq a^2 \} \).

3. Compute the volume of the domain \( U \subset \mathbb{R}^3 \) defined by
   
   a) \( U = \{ x \geq 0, \ y \geq 0, \ 1 \leq xy \leq 2, \ x \leq y \leq 2x, \ 0 \leq z \leq x + y \} \).

   b) \( U = \left\{ \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} \leq 1, \ \frac{x^2}{a^2} + \frac{y^2}{a^2} \leq \frac{z^2}{c^2}, \ z > 0 \right\} \).

4. Compute the integral of a differential 2-form:
   \[ \int_Q \left| \cos(x + y) \right| dx \wedge dy, \]
where the square $Q = \{0 \leq x, y \leq \pi\} \subset \mathbb{R}^2$ is endowed by the standard orientation of $\mathbb{R}^2$.

5. The Newton potential $V$ of a compact set $A \subset \mathbb{R}^3$ with the mass distribution given by the density function $\rho : U \to \mathbb{R}$ is defined at a point $x \notin A$ by the formula

$$
V(x) = \int_A \frac{\rho(y)}{||x - y||} dV_y = \iiint_A \frac{\rho(y_1, y_2, y_3) dy_1 dy_2 dy_3}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}}.
$$

Compute the Newton potential of the spherical layer

$$
A = \{R_1^2 \leq y_1^2 + y_2^2 + y_3^2 \leq R_2^2\}
$$

assuming that the density function $\rho \equiv 1$. Consider the cases when the point $x$ inside and outside of the spherical layer $A$.

6. Compute the volume of the $n$-dimensional simplex

$$
S_n = \left\{ x_1, \ldots, x_n \geq 0, \sum_{i=1}^n x_i \leq 1 \right\}.
$$

7. Let $0 < a < 1$ and $0 < b \leq 1$. Define a Cantor set $K_{a,b} \subset \mathbb{R}$ as follows. First define a sequence of closed subsets $[0, 1] = K_{a,b}^0 \subset K_{a,b}^1 \supset \ldots$ of an interval $[0, 1]$ inductively by the following procedure. Each of these sets is a finite union of disjoint closed subintervals of $[0, 1]$. If $K_{a,b}^n$ is already defined we then define $K_{a,b}^{n+1}$ by removing from each of the intervals $J$ forming $K_{a,b}^n$ an open subinterval centered at the center of $J$ of the length equal to the $ab^n|J|$, where $|J|$ is the length of the interval $J$.

Define

$$
K_{a,b} = \bigcap_{n=0}^{\infty} K_{a,b}^n.
$$

Prove that the set $K_{a,b}$ is Riemann measurable if and only if $b = 1$, and show that $\text{Vol}_1(K_{a,1}) = 0$.  

Each (sub)problem is 10 points.

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$K_{a,b}$ is measurable in the Lebesgue sense for all values of $a, b$, and it has positive Lebesgue measure if $b < 1$. 

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