Math 52H: Homework N2

Due to Friday, January 23

1. Do Exercise 4.5 from the online text:

   (a) For any special orthogonal operator (i.e. an orthogonal operator with determinant equal to 1) \( A : V \to V \) the operators \( A^* : \Lambda^k(V^*) \to \Lambda^k(V^*) \) and \( \ast \) commute, i.e.
   \[
   A^* \circ \ast = \ast \circ A^*.
   \]

   (b) Let \( A \) be a special orthogonal matrix of order \( n \) with \( \det A = 1 \). Prove that the absolute value of each \( k \)-minor \( M \) of \( A \) is equal to the absolute value of its complementary minor of order \( (n - k) \). Here \( k \) is any integer between 1 and \( n \). (Hint: Apply (a) to the form \( x_{i_1} \wedge \cdots \wedge x_{i_k} \)).

2. Given a parallelepiped \( P(v_1, v_2, v_3, v_4) \subset \mathbb{R}^4 \), compute the 3-dimensional volume of each of its 3-dimensional face. Here

   \[
   v_1 = (1, 1, 1, 1),
   v_2 = (1 - 1, 1, 1),
   v_3 = (1, 1, -1, 1),
   v_4 = (1, 1, 1, -1).
   \]

   Compare the orientation of \( \mathbb{R}^4 \) given by the basis \( v_1, v_2, v_3, v_4 \) with the orientation given by its standard basis \( e_1, e_2, e_3, e_4 \).
3. A vector subspace \( L \subset V \) of a vector space \( V \) is called \textit{invariant} with respect to a linear operator \( A : V \to V \) if \( A(v) \in L \) for each vector \( v \in L \).

Let \( A : V \to V \) be a linear operator, and \( l_1, \ldots, l_k \in V^* \) be linear independent vectors from the dual space \( V^* \). Suppose that
\[
A^*(l_1 \wedge \cdots \wedge l_k) = cl_1 \wedge \cdots \wedge l_k,
\]
for some non-zero real number \( c \in \mathbb{R} \). Prove that the vector subspace \( \text{Span}(l_1, \ldots, l_k) \) is invariant with respect to the dual operator \( A^* : V^* \to V^* \).

4. Let \( \eta \) be an exterior 2-form on a vector space \( V \). Suppose that \( \eta \wedge \eta = 0 \). Prove that there exists two 1-forms \( \alpha, \beta \in V^* \) such that \( \eta = \alpha \wedge \beta \).

5. An exterior 2-form \( \beta \) on \( \mathbb{R}^4 \) is called self-dual if \( \star \beta = \beta \). What is the dimension of the space of self-dual exterior 2-forms on \( \mathbb{R}^4 \). Find a basis of this space.

All problems and subproblems are 10 points.