52H HW 6 solutions

1. We calculate

\[ I := \int_D \left| \frac{x+y}{2} - r^2 \right| dV = \int_0^{2\pi} \int_0^1 r^2 \left| \frac{\cos \theta + \sin \theta}{2} - r \right| dr d\theta \]

\[ = \int_0^{2\pi} \int_0^1 r^2 \frac{\sin(\theta + \pi/4)}{\sqrt{2}} - r dr d\theta \]

\[ = \int_0^{2\pi} \int_0^1 r^2 \sin \theta / \sqrt{2} - r dr d\theta. \]

The quantity inside the absolute values is negative for all \( r \) if \( \theta \in [\pi, 2\pi] \). If \( \theta \in [0, \pi] \), we break the \( r \)-integral into two parts:

\[ I = \int_0^{\pi} \int_0^1 r^2 \frac{\sin \theta}{\sqrt{2}} - r dr d\theta + \int_0^{\pi} \int_0^1 r^2 \left( r - \frac{\sin \theta}{\sqrt{2}} \right) dr d\theta \]

\[ = \frac{\pi}{2} + \frac{1}{24} \int_0^{\pi} \sin^4 \theta d\theta \]

\[ = \frac{\pi}{2} + \frac{1}{24} \int_0^{\pi} \frac{1}{2} (1 - \cos 2\theta) - \frac{3}{4} \sin^2 2\theta d\theta \]

\[ = \frac{33\pi}{64}. \]

2. Consider the map \( A(x) = -x \) on \( R^2 \). Then \( A(D_a) = D_a \) and \( JA = 1 \). Therefore

\[ \int_{D_a} x^{20} y^{15} dV = \int_{A(D_a)} x^{20} y^{15} dV \]

\[ = \int_{D_a} (Ax)^{20} (Ay)^{15} |JA| dV \]

\[ = - \int_{D_a} x^{20} y^{15} dV. \]

We deduce the integral is zero.

3. Let \( R \) be the domain \( R = \{(x, y) : x \geq 0, y \geq 0, 1 \leq xy \leq 2, x \leq y \leq 2x\} \) in \( R^2 \). Then

\[ \text{Vol}(U) = \int_R x + y dV_{R^2}. \]

Define the change of coordinates \( (u, v) := \phi(x, y) = (xy, y/x) \), so that \( \phi(R) = [1, 2] \times [1, 2] \). This has inverse

\[ (x, y) = \phi^{-1}(u, v) = (\sqrt{u/v}, \sqrt{uv}). \]
We calculate
\[ J(\phi^{-1}) = \left| \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial z}{\partial u} \right| = \frac{1}{4} \left| \frac{1}{\sqrt{uv}} - \frac{\sqrt{u/v}}{\sqrt{v/y}} \right| = \frac{1}{2v}. \]

(here this is the absolute value of the determinant, as we consider unoriented volume).

Therefore
\[ \text{Vol}(U) = \int_{\phi R} (x + y) \circ \phi^{-1} |J \phi^{-1}|\,dV_{R^3} \]
\[ = \int_1^2 \int_1^2 \left( \frac{u}{v} + \sqrt{uv} \right) \frac{1}{2} |\,dudv \]
\[ = \int_1^2 \sqrt{u}/2du \int_1^2 v^{-3/2} + v^{-1/2}dv \]
\[ = \frac{4 - \sqrt{2}}{3}. \]

B. Introduce weighted polar coordinates
\[ x = ar \sin \phi \cos \theta \]
\[ y = ar \sin \phi \sin \theta \]
\[ z = cr \cos \phi, \]
so that \( U \) becomes the region defined by
\[ 0 \leq r \leq 1, 0 \leq \phi \leq \pi/4, -\pi \leq \theta \leq \pi. \]

The Jacobian of the map \((r, \phi, \theta) \rightarrow (x, y, z)\) is
\[ J = \begin{vmatrix} a \cos \theta \sin \phi & ar \cos \theta \cos \phi & -ar \sin \theta \sin \phi \\ a \sin \theta \sin \phi & ar \sin \theta \cos \phi & ar \cos \theta \sin \phi \\ c \cos \phi & -cr \sin \phi & 0 \end{vmatrix} = a^2 cr^2 \sin \phi. \]

So
\[ \text{Vol}(U) = \int_U \,dV_{R^3} \]
\[ = \int_0^1 \int_{\pi/4}^{\pi/4} \int_{-\pi}^\pi a^2 cr^2 \sin \phi d\theta d\phi dr \]
\[ = 2\pi a^2 c \frac{1}{3} \int_0^{\pi/4} \sin \phi d\phi \]
\[ = \frac{(2 - \sqrt{2})\pi}{3} a^2 c. \]
4. Since \( \cos((\pi - x) + (\pi - y)) = \cos(x + y) \), we have that
\[
\int_Q |\cos(x + y)| dV
= 2 \int_{Q \cap \{x + y \leq \pi\}} |\cos(x + y)| dV
= 2 \int_{Q \cap \{x + y \leq \pi/2\}} \cos(x + y) dV - 2 \int_{Q \cap \{\pi/2 \leq x + y \leq \pi\}} \cos(x + y) dV
= 2 \int_0^{\pi/2} \int_{-x}^{\pi/2-x} \cos(x + y) dy dx - 2 \int_0^\pi \int_{\text{max}(\pi/2-x,0)}^{\pi-x} \cos(x + y) dy dx
= \pi - \int_0^{\pi/2} \sin x dx + \pi + \int_0^{\pi/2} \sin x dx
= 2\pi.
\]

5. Without loss of generality we can take \( x = \ell e_3 \), with \( \ell > 0 \). In polar coordinates
\[
(y_1, y_2, y_3) = (r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi)
\]
we get that
\[
|x - y|^2 = \ell^2 + r^2 - 2 \ell r \cos \phi.
\]
Therefore
\[
V(\ell e_3) = \int_{R_1}^{R_2} \int_0^{2\pi} \int_0^\pi \frac{r^2 \sin \phi}{\sqrt{r^2 + \ell^2 - 2r \ell \cos \phi}} d\theta d\phi dr
= 2\pi \int_{R_1}^{R_2} \frac{r}{\ell} \left[ \sqrt{r^2 + \ell^2 + 2r \ell} - \sqrt{r^2 + \ell^2 - 2r \ell} \right] dr
= \frac{2\pi}{\ell} \int_{R_1}^{R_2} r(\ell + r - |\ell - r|) dr.
\]
So if \( x \) lies outside the outer shell (i.e. \( \ell > R_2 \))
\[
V(x) = \frac{4\pi}{3} (R_2^3 - R_1^3) \frac{1}{|x|},
\]
whereas if \( x \) lies inside the inner shell (so \( \ell < R_1 \))
\[
V(x) = 2\pi (R_2^2 - R_1^2).
\]
6. Trivially Vol₁(S₁) = Vol₁([0, 1]) = 1. We calculate Volₙ(Sₙ) by induction:

\[
\text{Vol}_n(S_n) = \int_0^1 \text{Vol}_{n-1}((1 - x_1)S_{n-1}) \, dx_1 = \sum_{i=0}^{n-1} \left( \frac{1}{n} \right) \cdot \int_0^1 (1 - x_1)^{n-1} \, dx_1 = \frac{1}{n!},
\]

7. Denote |A| = Vol₁(A) for any \( A \subset \mathbb{R} \), and write \( K = \bigcap_n K^n \). As a finite union of intervals, each \( K^n \) is Riemann-measurable. We have that

\[
|K^n| = (1 - ab^{n-1})|K^{n-1}| = \prod_{k=0}^{n-1} (1 - ab^k).
\]

Let \( \ell = \lim_{n \to \infty} |K^n| \), which exists because the \( K^n \) are nested. If \( \ell = 0 \) then \( K \) is Riemann-measurable and has volume 0 by Proposition 9.16(3) in the notes. We show \( K \) is not Riemann-measurable if \( \ell \neq 0 \).

Let \( \mathcal{P} = \{P_i\} \) be a partition of \([0, 1] \). I claim that \( L(\chi_K, \mathcal{P}) = 0 \). Suppose, towards a contradiction, that \([x, y] \subset K \). Then for each \( n \), \([x, y] \subset J_n \) for some (maximal) closed interval \( J_n \subset K^n \). But

\[
|J_n| = \frac{1}{2} (1 - ab^{n-1})|J_{n-1}| \leq 2^{-n}
\]

is smaller than \( y - x \) for large \( n \). This is a contradiction, proving the Claim.

If \( P_i \cap K = \emptyset \), then \( P_i \cap K^n = \emptyset \) for some \( n \). Otherwise, we could choose a sequence \( x_n \in P_i \cap K^n \). By compactness of \( P_i \), a subsequence \( x_{n'} \) converges to some \( x \in P_i \), but then \( x \in K \) since the \( K^n \) are closed, nested.

So we have an \( n \) with \( P_i \cap K^n = \emptyset \) for every \( P_i \in \mathcal{P} \) that doesn’t meet \( K \). This implies

\[
\sum_{i : P_i \cap K = \emptyset} |P_i| \leq 1 - |K^n|
\]

and therefore

\[
U(\chi_K, \mathcal{P}) = \sum_{i : P_i \cap K \neq \emptyset} |P_i| \geq |K^n| \geq \ell.
\]

This proves \( K \) is Riemann-measurable (with volume 0) iff \( \ell = 0 \). It remains to show \( \ell = 0 \) iff \( b = 1 \). Since

\[
|K^n| = \exp \left( \sum_{k=0}^{n-1} \log(1 - ab^k) \right)
\]

it suffices to show \( \lim_{n \to \infty} \sum_{k=0}^{n-1} \log(1 - ab^k) = -\infty \) iff \( b = 1 \).
If $b = 1$, then

$$\sum_{k=0}^{n-1} \log(1 - ab^k) = n \log(1 - a) \rightarrow -\infty \text{ as } n \rightarrow \infty,$$

since $a < 1$. Suppose now $b < 1$. We have (for $0 < x < 1$)

$$\log(1 - x) = \sum_{k=1}^{\infty} -\frac{x^k}{k} \geq -x$$

and therefore

$$\sum_{k=0}^{n-1} \log(1 - ab^k) \geq -a \sum_{k=0}^{n-1} b^k \geq -\frac{a}{1 - b}$$

for every $n$. 