1. Let \( f = \sum_{i=1}^{n} x_{i} x_{i+n} \) and let \( dV = dx_{1} \wedge \cdots \wedge dx_{2n} \) denote the volume form. We have
\[
d\Omega = \frac{1}{f^n} d(\omega \wedge \theta) - \frac{n}{f^{n+1}} df \wedge \omega \wedge \theta \\
= \frac{n}{f^n} \left( dV - \frac{1}{f} \sum_{i=1}^{n} (x_{n+i} dx^i + x_i dx^{n+i}) \omega \wedge \theta \right) \\
= \frac{n}{f^n} \left( dV - \frac{1}{f} \sum_{i=1}^{n} (x_i x_{n+i}) dV \right) \\
= 0.
\]

2. Note that \( \theta_k = r^{-k} \sum_i x_i \star dx^i \). We have
\[
d\theta_k = \frac{-k}{r^{k+1}} \sum_i x_i dr \wedge \star dx_i + \frac{1}{r^k} \sum_{i,j} \partial_j x_i dx^j \wedge \star dx_i \\
= \frac{-k}{r^{k+1}} \sum_{i,j} x_i x_j \frac{1}{r} dx^j \wedge \star dx_i + \frac{n}{r^k} dV \\
= \frac{n-k}{r^k} dV
\]
is 0 if and only if \( n = k \).

3. By virtue of being closed on \( R^3 \) we know \( \omega = df \) by Stokes theorem, where \( f(x) = \int_{0}^{x} \omega \) is the integral of \( \omega \) along any path \( \gamma \) from 0 to \( x \). In particular, choosing \( \gamma(t) = tx \), we get that
\[
f(x) = \int_{0}^{1} \sum_i F_i(tx) x_i dt \\
= \sum_i F_i(x) x_i \int_{0}^{1} t dt \\
= \sum_i x_i F_i(x) \frac{1}{2}.
\]

We show explicitly that \( \omega = df \), avoiding the use of Stokes theorem. By virtue of \( d\omega = 0 \) we obtain that
\[
\partial_i F_j - \partial_j F_i = 0
\]
for each \( i < j \). By homogeneity of each \( F_i \) we can calculate the total \( t \)-derivative two separate ways, using the chain rule to deduce (for each \( k \))
\[
\frac{d}{dt}|_{t=1} F_k(tx) = \sum_i x_i \partial_i F_k(x) \\
= F_k(x).
\]
Using the above two expression, we now calculate

\[ df = \frac{1}{2} \sum_{i,j} \partial_i (x_j F_j) dx^i \]

\[ = \frac{1}{2} \sum_{i,j} (\delta_{ij} F_j + x_j \partial_i F_j) dx^i \]

\[ = \frac{1}{2} \left( \sum_i F_i dx^i + \sum_i x_j \partial_j F_i dx^i \right) \]

\[ = \omega. \]

4. A. If \( \omega \in \Omega^k \), then \( d^* \omega \in \Omega^{n-k+1} \). We calculate

\[ (-1)^k \star^{-1} d^* \omega = (-1)^{k+(n-k+1)(k-1)} \star d^* \omega \]

\[ = (-1)^{n(k-n-1)+2k-k(k-1)} \star d^* \omega \]

\[ = (-1)^{n(k+n+1)} \star d^* \omega \]

since \( k(k-1) \) is always even, and \( (-1)^{-1} = -1 \).

B. To eliminate notational confusion we will write \( \delta \) instead of \( \partial \) as the adjoint of \( d \), since we will write \( \partial_i \equiv \frac{\partial}{\partial x_i} \) for the usual partial derivative. Now \( \Delta = d^2 + d\delta + \delta d + \delta^2 \), and

\[ \delta^2 = (-1)^{k-1} \star^{-1} d \star (-1)^k \star^{-1} d \star = - \star^{-1} d^2 \star = 0, \]

so in fact \( \Delta = d\delta + \delta d \) is an operator \( \Omega^k \to \Omega^k \).

For a function \( f \), we calculate

\[ \Delta f = \delta df \]

\[ = - \star^{-1} d \sum_i \partial_i f \star dx^i \]

\[ = - \star^{-1} \sum_{i,j} \partial_j \partial_i f dx^j \wedge \star dx^i \]

\[ = - \star^{-1} \sum_{i,j} \delta_{ij} \partial_i \partial_j f dV \]

\[ = - \sum_i \partial_i^2 f \]

\[ = - \Delta f. \]

Where we define the operator \( \tilde{\Delta} \) to act on functions by \( \tilde{\Delta} f = \sum_i \partial_i \partial_i f \). (\( \tilde{\Delta} \) is the "connection laplacian" of Euclidean space, while \( \Delta \) is the "Hodge laplacian".)

Note that for a general metric it is not true that \( d(\star dx^i) = 0 \). We used that in Euclidean space \( \star dx^i = (-1)^{i-1} dx^1 \wedge \cdots \hat{dx}^i \cdots \wedge dx^n \), with that hat denoting omission.
For a 1-form \( \alpha = \sum_i \alpha_i dx^i \), we break the calculation into two parts. First,

\[
d\delta \alpha = - d \star^{-1} d \star \alpha
= - d \star^{-1} \sum_{i,j} \partial_j \alpha_i dx^j \wedge \star dx^i
= - d \left( \sum_i \partial_i \alpha_i \right)
= - \sum_{i,j} \partial_j \partial_i \alpha_i dx^j.
\]

We calculate

\[
\delta d \alpha = (-1)^2 \star^{-1} d \star d \alpha
= \star^{-1} d \sum_{i,j} \partial_i \alpha_j \star dx^i \wedge dx^j
= \star^{-1} \sum_{i,j,k} \partial_k \partial_i \alpha_j dx^k \wedge \star(dx^i \wedge dx^j)
= \star^{-1} \sum_{i,j} \left( \partial_i \partial_j \alpha_j dx^i \wedge \star(dx^i \wedge dx^j) + \partial_j \partial_i \alpha_j dx^j \wedge \star(dx^i \wedge dx^j) \right).
\]

Now for any \( i, j, k \) (with \( i \neq j \)) we have

\[
dx^k \wedge dx^j \wedge \star(dx^i \wedge dx^j) = \langle dx^k \wedge dx^j, dx^i \wedge dx^j \rangle > dV
= \begin{cases} 
  dV & \text{if } k = i \\
  0 & \text{otherwise}
\end{cases},
\]

and so \( dx^j \wedge \star(dx^i \wedge dx^j) = \star dx^i \). And of course \( dx^i \wedge \star(dx^i \wedge dx^j) = - \star dx^j \).

We continue our second calculation:

\[
\delta d \alpha = \star^{-1} \sum_{i,j} \left( - \partial_i \partial_j \alpha_j \star dx^j + \partial_j \partial_i \alpha_j \star dx^i \right)
= \sum_{i,j} \left( - \partial_i \partial_j \alpha_j dx^j + \partial_j \partial_i \alpha_j dx^i \right).
\]

We deduce that

\[
\Delta \alpha = - \sum_i \left( \sum_j \partial_j \partial_i \alpha_j \right) dx^i = - \sum_i (\Delta_i) dx^i.
\]

I claim that \( \Delta \star = \star \Delta \). To prove this, consider

\[
\star \Delta \star = \star (\delta d + d \delta) \star
= \star ((-1)^{n-k+1} \star^{-1} d \star d + (-1)^{n-k} d \star^{-1} d \star) \star
= (-1)^{n+k+1} d \star d \star + (-1)^{n-k+k(n-k)} d \star^{-1} d \star
= (-1)^{n+k+1+n} d \delta + (-1)^{n+k+k(n-k)+k+1} d \star d \star
= (-1)^{k(n-k)} d \delta + (-1)^{k+1+n(k+1)+n+1} \delta \star d \star
= (-1)^{k(n-k)} \Delta,
\]
using several times that \(-1 \equiv 1 \mod 2\) and \(k \equiv k^2 \mod 2\). This proves the Claim.

So if \(\tau = f dV = \star f\) is a top-dimensional form, then

\[
\Delta \tau = \star \Delta f = (-\Delta f) dV.
\]