Math 52H: Solutions to Midterm Exam

February 2, 2011

1. In the standard Euclidean $\mathbb{R}^{2n}$ with Cartesian coordinates $x_1, y_1, \ldots, x_n, y_n$ consider an exterior 2-form

$$\eta = \sum_{k=1}^{n} x_k \wedge y_k.$$

Given a 1-form $\alpha = \sum_1^n a_i x_i + \sum_1^n b_j y_j$, where $a_i, b_j \in \mathbb{R}$ are constants, compute the 1-form

$$\beta = \star\left(\alpha \wedge \underbrace{\eta \wedge \cdots \wedge \eta}_{n-1}\right).$$

We have

$$\eta^{n-1} = (n-1)! \sum_1^n x_1 \wedge \overset{2j}{\check{\ldots}} \wedge y_n \;\; (x_j \wedge y_j \text{ is missing}).$$

Then

$$x_j \wedge \eta^{n-1} = (n-1)! \sum_1^n x_1 \wedge \overset{j}{\check{\ldots}} \wedge y_n \;\; (y_j \text{ is missing})$$

and

$$x_{2j} \wedge \eta^{n-1} = (n-1)! \sum_1^n x_1 \wedge \overset{j}{\check{\ldots}} \wedge x_{2n} \;\; (x_j \text{ is missing}).$$

Hence, $\star(x_j \wedge \eta^{n-1}) = (n-1)! y_j$ and $\star(y_j \wedge \eta^{n-1}) = -(n-1)! x_j$.

Therefore

$$\beta = \star(\alpha \wedge \eta^{n-1}) = \sum_1^n a_j \star \big((x_j \wedge \eta^{n-1}) + b_j \star (y_j \wedge \eta^{n-1})\big)$$
$$= (n-1)! \sum_1^n (a_j y_j - b_j x_j).$$

Note that these formulas also holds for $n = 1$. In this case, $\star(a_1 x_1 + b_1 y_1) = a_1 y_1 - b_1 x_1$.

2. Denote coordinates in $\mathbb{R}^2$ by (u, s) and in $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4). Consider a differential 1-form

$$\alpha := x_4(dx_3 - x_2 dx_1).$$

Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $\frac{\partial h}{\partial s}(u, s) > 0$ for all $(u, s) \in \mathbb{R}^2$. Find a smooth function $g : \mathbb{R}^2 \to \mathbb{R}$ such that the form $G^*\alpha$ on $\mathbb{R}^2$ is closed for the map $G : \mathbb{R}^2 \to \mathbb{R}^4$ given by the formula

$$G(u, s) = \left(u, \frac{\partial h}{\partial u}(u, s), h(u, s), g(u, s)\right).$$

Note that there are infinitely many functions g which satisfy this condition. You need to find just any of them.

We have

$$G^*\alpha = g\left(dh - \frac{\partial h}{\partial u} du\right) = g\left(\frac{\partial h}{\partial u} du + \frac{\partial h}{\partial s} ds - \frac{\partial h}{\partial u} du\right) = g\left(\frac{\partial h}{\partial s} ds\right).$$

By assumption, $\frac{\partial h}{\partial s} > 0$. Define

$$g := \frac{1}{\frac{\partial h}{\partial s}}.$$

Then $G^*\alpha = ds$, and hence, $dG^*\alpha = dds = 0$.

4. In $\mathbb{R}^n \setminus 0$ consider the unit vector field in the radial direction

$$\mathbf{e}_r = \frac{1}{r} \sum_1^n x_i \frac{\partial}{\partial x_i},$$

where $r = \sqrt{\sum_1^n x_i^2}$. Denote $\eta := dx_1 \wedge \cdots \wedge dx_n$. Find all integer values $k \in \mathbb{Z}$ for which the differential $(n-1)$-form

$$\frac{1}{r^k} \mathbf{e}_r \lrcorner \eta \quad \text{is closed.}$$

Denote

$$\theta_k := \frac{1}{r^k}\mathbf{e}_r \lrcorner \eta.$$

We have

$$\frac{1}{r^k}\mathbf{e}_r \lrcorner \eta = \frac{1}{r^{k+1}}\sum_1^n (-1)^{j-1} x_j dx_1 \wedge \overset{j}{\check{\dots}} \wedge dx_n.$$

$$d\left(\frac{x_i}{r^{k+1}}\right) = \frac{dx_i}{r^{k+1}} - kx_i \frac{\sum_1^n x_j dx_j}{r^{k+3}}.$$

Hence

$$d\theta_k = \sum_{i=1}^n (-1)^{i-1} d\left(\frac{x_i}{r^{k+1}}\right) dx_1 \wedge \overset{i}{\check{\dots}} \wedge dx_n =$$

$$\left(\frac{n}{r^{k+1}} - (k+1)\frac{\sum_1^n x_i^2}{r^{k+3}}\right) dx_1 \wedge \dots \wedge dx_n$$

$$= \frac{n-k-1}{r^{k+1}} dx_1 \wedge \dots \wedge dx_n.$$

Hence, θ_k is closed if (and only if) $k = n-1$.

4. Let $D = \{x^2 + y^2 \le 1\} \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}^2$ a smooth map such that

- $f(x, y) = (x, y)$ if $\frac{1}{2} \le x^2 + y^2 \le 1$ and
- $f(0, 0) = (2, 2013)$.

Prove that there exists a point $a = (x_0, y_0) \in D$ such that the 2-form $f^*(dx \wedge dy)$ vanishes at a.

Denote the coordinate functions of f by f_1 and f_2. We have

$$f^*(dx \wedge dy) = \det Df\, dx \wedge dy = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}.$$

We need to show that $\det Df$ vanishes at some point $a \in D$ The continuous function f_1 achieves its maximum value at a poin $a \in D$. At the center of the ball it takes the value 2, while at the boundary of the ball D its values belong to $[-1, 1]$. Hence, a should be the interior point of D, and hence a critical point of f_1. Therefore, both partial derivatives of f_1 vanish at that point:

$$\frac{\partial f_1}{\partial x}(a) = 0, \ \frac{\partial f_1}{\partial y}(a) = 0.$$

Hence, the Jacobi matrix has its first column equal to zero at the point a, and therefore $\det Df(a) = 0$.