

Math 52H: Solutions to practice problems for the midterm

1. Consider the Euclidean space $V = \mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$ and the standard dot-product. The space $\Lambda(V^*)$ of all exterior k -forms for all $k = 0, \dots, 2n$ is also an Euclidean space with the scalar product of a k -form α and an l -form β defined by the formula

$$\langle\langle \alpha, \beta \rangle\rangle = \begin{cases} \star^{-1}(\alpha \wedge \star\beta), & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases}$$

Consider a linear operator $\Omega : \Lambda(V^*) \rightarrow \Lambda(V^*)$ defined by the formula $\Omega(\alpha) = \alpha \wedge \omega$, where $\omega = \sum_1^n x_i \wedge y_i$. Find the adjoint linear operator Ω^* , i.e. the operator $\Omega^* : \Lambda(V^*) \rightarrow \Lambda(V^*)$ such that

$$\langle\langle \Omega(\alpha), \beta \rangle\rangle = \langle\langle \alpha, \Omega^*(\beta) \rangle\rangle$$

for any forms $\alpha, \beta \in \Lambda(V^*)$.

We have

$$\langle\langle \Omega(\alpha), \beta \rangle\rangle = \star^{-1}(\alpha \wedge \omega \wedge \star\beta) = \star^{-1}(\alpha \wedge \star(\star^{-1}(\omega \wedge \star\beta))) = \langle\langle \alpha, \star^{-1}(\omega \wedge \star\beta) \rangle\rangle.$$

Therefore,

$$\langle\langle \alpha, \star^{-1}(\omega \wedge \star\beta) - \Omega^*(\beta) \rangle\rangle = 0$$

for any k -forms α, β , which means that

$$\Omega^*(\beta) = \star^{-1}(\omega \wedge \star\beta)$$

2. Consider two differential 1-forms in \mathbb{R}^3 :

$$\alpha = dx + ydz \quad \text{and} \quad \beta = xdy.$$

Prove that there is no map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $f^*(\beta) = \alpha$.

We have $d\alpha = dy \wedge dz$ and $d\beta = dx \wedge dy$. Thus, $\alpha \wedge \beta = dx \wedge dy \wedge dz$ and $\beta \wedge d\beta = 0$.

If $f^*(\beta) = \alpha$ then

$$f^*(\beta \wedge d\beta) = f^*\beta \wedge df^*\beta = \alpha \wedge d\alpha.$$

But $\beta \wedge d\beta = 0$, and this would imply that $\alpha \wedge d\alpha = 0$, which is a contradiction.

3. The cylindrical coordinates

$$r \in [0, \infty), \varphi \in [0, 2\pi), z \in \mathbb{R},$$

are introduced in \mathbb{R}^3 by the formulas

$$x = r \cos \varphi, y = r \sin \varphi, z,$$

where (x, y, z) are Cartesian coordinates. Consider a differential 1-form

$$\alpha = \cos r dz + \frac{r \sin r}{\pi} d\varphi.$$

Suppose that a curve $\Gamma \subset \mathbb{R}^3$ be given by the parametric equations

$$r = \frac{\pi}{4}, z = h(t), \varphi = 2t, t \in [0, \pi].$$

Find the function h such that $\alpha|_{\Gamma} = 0$ and $h(0) = 1$.

The restriction of α to Γ is equal $\left(\frac{\sqrt{2}}{2}h' + \frac{\sqrt{2}}{4}\right) dt$. Hence we get $h' = -\frac{1}{2}$, and thus $h(t) = -\frac{t}{2} + C$. The constant C is equal to $h(0) = 1$. Hence, $h(t) = -\frac{t}{2} + 1$.

4. Consider a smooth function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Let S_f be a surface in \mathbb{R}^4 given by equations

$$x_3 = \frac{\partial f}{\partial x_1}(x_1, x_2), \quad x_4 = \frac{\partial f}{\partial x_2}(x_1, x_2) \tag{1}$$

Suppose that this system of equations can be solved with respect to the coordinates x_2 and x_4 , i.e. there exist smooth functions $x_2 = g(x_1, x_3)$ and $x_4 = h(x_1, x_3)$ such that

$$\begin{aligned} x_3 &\equiv \frac{\partial f}{\partial x_1}(x_1, g(x_1, x_3)), \\ h(x_1, x_3) &\equiv \frac{\partial f}{\partial x_2}(x_1, g(x_1, x_3)). \end{aligned} \tag{2}$$

Prove that the Jacobian of the map $(h, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is equal to -1 , i.e. that

$$\begin{vmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_3} \end{vmatrix} = -1.$$

One can check that $\omega|_{S_f} = 0$. Let F be the map described in the Hint. Then we have

$$0 = F^*\omega = dx_1 \wedge dx_3 + dg \wedge dh = (1 + J)dx_1 \wedge dx_3,$$

where $J = \begin{vmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_3} \end{vmatrix}$. Hence, $J = -1$.

5. Consider a smooth differential k -form

$$\alpha = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k}(x) dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

in \mathbb{R}^n such that $f_{i_1 \dots i_k}(0) = 0$ (i.e. all coefficients of the form α are equal to 0 at the origin). Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the dilatation $x \mapsto 2x$. Suppose that $F^*\alpha = \alpha$. Prove that $\alpha \equiv 0$.

We have for any point $a \in \mathbb{R}^n$ and vectors $X_1, \dots, X_n \in \mathbb{R}_a^n$

$$\alpha_a(X_1, \dots, X_n) = (F^{-1})^* \alpha_a(X_1, \dots, X_n) = \alpha_{\frac{a}{2}} \left(\frac{1}{2}X_1, \dots, \frac{1}{2}X_n \right).$$

Iterating this formula, we get

$$\alpha_a(X_1, \dots, X_n) = (f^{-n})^* \alpha_a(X_1, \dots, X_n) = \alpha_{\frac{a}{2^n}} \left(\frac{1}{2^n}X_1, \dots, \frac{1}{2^n}X_n \right) \xrightarrow{n \rightarrow \infty} 0.$$

6. Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, consider a map $F : \mathbb{R}^n \rightarrow \mathbb{R}^{2n+1}$ defined by the formula

$$F(x_1, \dots, x_n) = \left(x_1, \dots, x_n, \frac{\partial f}{\partial x_1}(x_1, \dots, x_n), \dots, \frac{\partial f}{\partial x_n}(x_1, \dots, x_n), f(x_1, \dots, x_n) \right).$$

Compute $F^*(\alpha)$, where

$$\alpha = dx_{2n+1} - \sum_{i=1}^n x_{i+n} dx_i.$$

We have

$$F^* \alpha = df - \sum_1^n \frac{\partial f}{\partial x_i} dx_i = 0.$$