Math 52H: Homework N9

Due to Thursday, March 14

1. Prove that if S is a closed surface in $\mathbb{R}^3$, $\mathbf{n}$ its unit normal vector field and $\mathbf{v}$ any fixed vector then

$$\iint_S \langle \mathbf{n}, \mathbf{v} \rangle dS = 0.$$

2. Given a function $u : U \to \mathbb{R}$, where U is an open domain in $\mathbb{R}^n$ we denote by Δu the *Laplace operator*

$$\Delta u = \sum_1^n \frac{\partial^2 u}{\partial x_j^2}.$$

A function u is called *harmonic* in U if $\Delta u = 0$. Suppose that $n = 2$, i.e. U is a planar domain.

a) Prove that u is harmonic in U if and if for any closed 1-dimensional submanifold $\Gamma \subset U$ one has

$$\oint_\Gamma \frac{\partial u}{\partial \mathbf{n}} ds = 0,$$

where n is a unit normal vector field to Γ and $\frac{\partial u}{\partial \mathbf{n}} = du(\mathbf{n})$ is the directional derivative.

b) Prove that for any C^2-smooth function $u : U \to \mathbb{R}$ one has

$$\iint_S \left(\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 \right) dx_1 dx_2 = -\int_S u \Delta u \, dx_1 dx_2 + \oint_\Gamma u \frac{\partial u}{\partial \mathbf{n}} ds,$$

1

where $S \subset U$ is any compact domain with boundary Γ.

c) Let S and Γ be as in the previous problem. Prove that for any two C^2-functions $u, v : U \to \mathbb{R}$ one has the following identity:

$$\iint_S \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx_1 dx_2 = \oint_\Gamma \begin{vmatrix} \frac{\partial u}{\partial \mathbf{n}} & \frac{\partial v}{\partial \mathbf{n}} \\ u & v \end{vmatrix} ds.$$

3. Consider in $\mathbb{R}^3$ a differential 1-form

$$\alpha = (x^2 - x + y + z - yz)dx + (y^2 + x - y + z - xz)dy + (z^2 + x + y - z - xy)dz.$$

Prove that α is exact and find its primitive, i.e. the function F such that $dF = \alpha$.

4. Given a vector field

$$\mathbf{v} = (x^2 + y^2 + z^2) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right),$$

compute $\text{Work}_\Gamma(\mathbf{v})$ where Γ is a curve in $\mathbb{R}^3$ connecting points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. The curve is oriented from A to B.

Each problem (including subproblems in 2) is 10 points.