Math 52H: Homework N8

Due to Friday, March 9

1. Compute

a)
$$\int_\gamma \frac{xdx + ydy + zdz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},$$

where the path $\gamma : [0, 1] \to \mathbb{R}^3$ is given by

$$\gamma(t) = (t^3 + 1, \sin\frac{\pi t}{2} + 1, 2^t).$$

b)
$$\int_\gamma \frac{xdy - ydx}{x^2 + y^2},$$

where $\gamma : [0, \pi] \to \mathbb{R}^2$ is given by

$$\gamma(t) = \left(1 + \frac{1}{2}\sin\frac{t^2}{\pi}, \cos\frac{t^3}{2\pi^2}\right).$$

2. Let $A \subset V$ be a closed oriented manifold of dimension $n = k + l + 1$, ω a differential k-form and η a differential l-form on V. Prove that

$$\int_A \omega \wedge d\eta = C \int_A \eta \wedge d\omega$$

for some constant C, and find C.

3. Let Γ be a closed oriented curve in $\mathbb{R}^2$ of length L. Prove that

$$\left| \int_\Gamma P dx + Q dy \right| \leq L \max_\Gamma \sqrt{P^2 + Q^2}.$$

4. Using Stokes' theorem compute the area of the domain in $\mathbb{R}^2$ bounded by the astroid

$$x = a \cos^3 t$$
$$y = b \sin^3 t,$$

$a, b > 0$, $0 \leq t \leq 2\pi$.

5. Suppose that the circle $S_R = \{x^2 + y^2 = R^2\} \subset \mathbb{R}^2$ is oriented counter-clockwise. Denote

$$I(R) := \int_{S_R} \frac{y dx - x dy}{(x^2 + xy + y^2)^2}.$$

Prove that $I(R) \underset{R \to \infty}{\longrightarrow} 0$.

6. Assuming that the Earth is a sphere of radius $R = 6.3 \cdot 10^6 m$ compute the area between the Tropic of Cancer and the Arctic Circle.

7. Using Stokes' theorem compute the integral

$$\int_C (y + z) dx + (z + x) dy + (x + y) dz),$$

where C is the ellipse, defined parametrically by the formulas

$$x = a \sin^2 t,$$
$$y = 2a \sin t \cos t,$$
$$z = a \cos^2 t,$$

$t \in [0, \pi]$. The ellipse is oriented by the parameter t.

Each problem (including subproblems in 1) is 10 points.