Math 52H: Homework N8

Due to Friday, March 1

1. Coordinate charts U_+ and U_- on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ are given by the stereographic projections from the South and North pole, respectively. Find explicitly the transition map between the two coordinate charts.

2. For the following manifolds define coordinate charts and transition maps between them:

a) projective space $\mathbb{R}P^n$, i.e. the space whose points are lines in $\mathbb{R}^{n+1}$ passing through the origin;

b) the orthogonal group $SO(3)$, i.e. the space of ortogonal 3×3-matrices with determinant $+1$;

c) the tangent bundle $T(S^2)$ to the 2-dimensional sphere, i.e. the space of pairs (X, Y), where $X \in S^2$ is a unit vector in $\mathbb{R}^3$, and $Y \in \mathbb{R}^3_X$ is a vector originated at X which is tangent to the sphere S^2 (or, equivalently, orthogonal to X).

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Consider a graphical n-dimensional submanifold with boundary

$$L_f = \left\{ (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}; \sum_1^n x_j^2 \leq 1; \;\; y_j = \frac{\partial f}{\partial x_j}(x_1, \dots, x_n), \; j = 1, \dots, n \right\} \subset \mathbb{R}^{2n}.$$

Suppose that L_f is parameterized by the map $F : D^n = \{\sum_1^n x_j^2 \leq 1\} \to L_f$ given by the formula

$$(x_1, \ldots, x_n) \stackrel{F}{\mapsto} \left(x_1, \ldots, x_n, \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n), \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n) \right).$$

Suppose that $n = 2k$. Compute

$$\int_{D^n} F^*(dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n)^{\wedge k}.$$

4. Let $S = \{x^2 + y^2 + z^2 = 1; z \geq 0\} \subset \mathbb{R}^3$. Use the stereographic coordinate chart given by the projection from the South pole to compute $\int_S dx \wedge dy + 2z dz \wedge dx$.

Each problem (and subproblems in Problem 2) is 10 points.