Math 52H: Homework N4

Due to Friday, February 8

1. Given polar coordinates (r, ϕ) in $\mathbb{R}^2$ we denote $\mathbf{e}_r := \frac{\partial}{\partial r}$, $\mathbf{e}_\phi := \frac{1}{r}\frac{\partial}{\partial \phi}$. Consider a vector field $\mathbf{v} = a\,\mathbf{e}_r + b\,\mathbf{e}_\phi$, where a, b are functions on $\mathbb{R}^2$. Compute $\operatorname{div} \mathbf{v}$.

2. Consider in $\mathbb{R}^{2n}$ differential forms

$$\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \text{ and } \theta = \sum_{n+1}^{2n} (-1)^{j-1} x_j dx_{n+1} \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{2n}.$$

Prove that the $(2n-1)$-form

$$\Omega = \frac{\omega \wedge \theta}{(\sum_1^n x_i x_{i+n})^n}$$

is closed.

3. Consider an operator $\partial = (-1)^k \star^{-1} d\star : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$. Note that we equivalently can write $\partial = (-1)^{nk+n+1} \star d\star$. Denote $\Delta := (\partial + d)^2$ (this is *Laplace-de Rham operator*). Verify that Δ is an operator $\Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$ and find explicit formulas for $\Delta\alpha$ for the cases when $n = 3$ and $k = 0, 1, 2, 3$. All computations should be done in the standard Cartesian coordinates in $\mathbb{R}^n$ with respect to the standard Euclidean stracture defined by the dot-product.

Each problem is 10 points.