Math 52H: Homework N3

Due to Friday, February 1

1. Given a parallelepiped $P(v_1, v_2, v_3, v_4) \subset \mathbb{R}^4$, compute the 3-dimensional volume of each of its 3-dimensional face. Here

$$v_1 = (1, 1, 1, 1),$$

$$v_2 = (1 - 1, 1, 1),$$

$$v_3 = (1, 1, -1, 1),$$

$$v_4 = (1, 1, 1, -1).$$

Compare the orientation of $\mathbb{R}^4$ given by the basis v_1, v_2, v_3, v_4 with the orientation given by its standard basis e_1, e_2, e_3, e_4.

2. A vector subspace $L \subset V$ of a vector space V is called *invariant* with respect to a linear operator $\mathcal{A} : V \to V$ if $\mathcal{A}(v) \in L$ for each vector $v \in L$.

Let $\mathcal{A} : V \to V$ be a linear operator, and $l_1, \ldots, l_k \in V^*$ be linear independent vectors from the dual space V^*. Suppose that

$$\mathcal{A}^*(l_1 \wedge \cdots \wedge l_k) = c\, l_1 \wedge \cdots \wedge l_k,$$

for some non-zero real number $c \in \mathbb{R}$. Prove that the vector subspace $\text{Span}(l_1, \ldots, l_k)$ is invariant with respect to the dual operator $\mathcal{A}^* : V^* \to V^*$.

3. Let $\mathcal{P}(n)$ denote the space of polynomials of $P(x_1, \ldots, x_n)$ of n variables (of arbitrary degree) with real coefficients. Let $D : P \to P$ be a linear operator which statisfies the

Leibniz rule:

$$D(fg) = D(f)g + fD(g), \text{ for any } f, g \in \mathcal{P}(n).$$

Prove that there exists a vector field $v = \sum_1^n P_i \frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$ with polynomial coefficients $P_i \in \mathcal{P}(n)$ such that $D = D_v$, i.e.

$$D(f) = df(v) = \sum_1^n P_i \frac{\partial f}{\partial x_i}$$

for any $f \in \mathcal{P}^n$.

4. In $\mathbb{R}^2$ with the standard dot-product consider a basis $v_1 = (1,0), v_2 = (1,1)$. Let (y_1, y_2) be coordinates dual to this basis. Given a function $f(y_1, y_2)$ compute its gradient in these coordinates, i.e. find functions $g_1(y_1, y_2), g_2(y_1, y_2)$ such that $\nabla f = g_1 \frac{\partial}{\partial y_1} + g_2 \frac{\partial}{\partial y_2}$.

5. Spherical coordinates $\rho \in [0, \infty)$, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi)$, are introduced in $\mathbb{R}^3$ by the formulas:

$$x = \rho \sin\varphi \cos\theta,\ y = \rho \sin\varphi \sin\theta, z = \rho\cos\varphi.$$

Express the 1-form $dz + \frac{1}{2}(xdy - ydx)$ in spherical coordinates.

6. Consider a closed differential 1-form $\omega = F_1 dx + F_2 dy + F_3 dz$ in $\mathbb{R}^3$. Suppose that each function F_k, $k = 1, 2, 3$, satisfies the homogeneity equation

$$F_k(tx, ty, tz) = tF_k(x, y, z),$$

for any $t \in \mathbb{R}$. Prove that $\omega = df$, where

$$f(x, y, z) = \frac{1}{2}\left(xF_1(x, y, z) + yF_2(x, y, z) + zF_3(x, y, z)\right).$$

All problems and subproblems are 10 points.