Math 52H: Solutions to the Final Exam

March 18, 2013

1. Let $T \subset \mathbb{R}^3$ be the torus defined by the parametric equations

$$x = (a + R\cos\theta)\cos\phi$$
$$y = (a + R\cos\theta)\sin\phi$$
$$z = R\sin\theta,$$

where $0 < R < a$ are constants. the parameters ϕ, θ take values in $[0, 2\pi]$

a) Compute the area of T;

b) Compute the volume of the domain bounded by the torus T.

a) Let us compute the pull-back of the area form σ of the torus T by the parametrization map

$$\Phi(\theta, \phi) = (a + R\cos\theta)\cos\phi, (a + R\cos\theta)\sin\phi, R\sin\theta).$$

We have

$$\frac{\partial \Phi}{\partial \theta} = (-R\sin\theta\cos\phi, -R\sin\theta\sin\phi, R\cos\theta),$$
$$\frac{\partial \Phi}{\partial \phi} = (-(a + R\cos\theta)\sin\phi, (a + R\cos\theta)\cos\phi, 0).$$

Then

$$E = \left\| \frac{\partial \Phi}{\partial \theta} \right\|^2 = R^2,$$
$$E = \left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \phi} \right\rangle = 0,$$
$$G = \left\| \frac{\partial \Phi}{\partial \phi} \right\|^2 = (a + R\cos\theta)^2.$$

Hence,

$$\sqrt{EG - F^2} = R(a + R\cos\theta),$$

and therefore,

$$\Phi^* \sigma = R(a + R\cos\theta) d\theta \wedge d\phi$$

and

$$\text{Area}(T) = \int_T \sigma = \int_{0 \le \phi, \theta \le 2\pi} \Phi^* \sigma = \int_0^{2\pi} \int_0^{2\pi} R(a + R\cos\theta) d\theta d\phi = 4\pi^2 aR.$$

b) Denote by U the domain bounded by T. By Stokes' theorem we have

$$\text{Vol}(U) = \int_U dx \wedge dy \wedge dz = \int_T z dx \wedge dy.$$

Using the parameterization Φ we get

$$\int_T z dx \wedge dy = \int_{0 \le \phi, \theta \le 2\pi} \Phi^*(z dx \wedge dy)$$
$$= R\sin\theta \left(-R\sin\theta\cos\phi d\theta - (a + R\cos\theta)\sin\phi d\phi\right) \wedge \left(-R\sin\theta\sin\phi d\theta + (a + r\cos\theta)\cos\phi) d\phi\right)$$
$$= R^2(a + R\cos\theta)\sin^2\theta d\phi \wedge d\theta.$$

Thus,

$$\text{Vol}(U) = \int_{0 \le \phi, \theta \le 2\pi} R^2(a + R\cos\theta)\sin^2\theta d\phi \wedge d\theta = 2\pi^2 aR^2.$$

2. Let u, v be two smooth functions on the unit disc $D = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$. Suppose that

- $u = x, v = y$ when $x^2 + y^2 \leq \frac{1}{2}$;
- $u^2 + v^2 \neq 0$ in $D \setminus 0$.

Compute

$$\int\limits_{\partial D} \frac{udv - vdu}{u^2 + v^2}.$$

Here ∂D is oriented counter-clockwise.

Consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2 0$ given by the formula

$$F(x, y) = (u(x, y), v(x, y)).\ (x, y) \in \mathbb{R}^2.$$

Then

$$\frac{udv - vdu}{u^2 + v^2} = F^*\alpha, \text{ where } \alpha = \frac{xdy - ydx}{x^2 + y^2}.$$

The form $F^*\alpha$ is closed in $\mathbb{R}^2 \setminus 0$ and hence, $\int\limits_{\partial D} F^*\alpha = \int\limits_{\partial \widetilde{D}} F^*\alpha$, where $\widetilde{D} = \{x^2 + y^2 \leq \frac{1}{2}\}$. But

$$\int\limits_{\partial \widetilde{D}} F^*\alpha = \int\limits_{\partial \widetilde{D}} \alpha = \int\limits_{\partial \widetilde{D}} d\phi = 2\pi.$$

3. Consider $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2). Denote $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a smooth function equal to x_2 outside the ball $B_1(0)$ of radius 1 centered at 0. Suppose a vector vield $\mathbf{v}$ satisfies

$$\mathbf{v} \lrcorner \omega = dH.$$

Compute the flux of $\mathbf{v}$ through the 3-dimensional disc

$$D = \{y_2 = 0,\ x_1^2 + y_1^2 + x_2^2 \leq 2\},$$

co-oriented by the normal vector $(0,0,0,1)$ at the origin.

$\mathrm{Flux}_D\, \mathbf{v} = \int\limits_D \eta$, where $\eta = \mathbf{v} \lrcorner \Omega$, $\Omega = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$. Let us compute η. we have

$$\eta = \mathbf{v} \lrcorner \Omega = \frac{1}{2}(\mathbf{v} \lrcorner \omega^2) = (\mathbf{v} \lrcorner \omega) \wedge \omega = dH \wedge \omega = d(H\omega).$$

Applying Stokes' theorem we find

$$\int\limits_D \eta = \int\limits_{\partial D} H\omega.$$

Recall that by assumption $H|_{\partial D} = x_2$. Using again Stokes' theorem we conclude that

$$\int\limits_{\partial D} H\omega = \int\limits_{\partial D} x_2\omega = \int\limits_D dx_2 \wedge \omega = \int\limits_D dx_1 \wedge dy_1 \wedge dx_2.$$

The absolute value of the latter integral is just the volume of the Euclidean ball of radius $\sqrt{2}$, i.e. it is equal to $\frac{8\pi\sqrt{2}}{3}$. However, the orientation of D is determined by the co-orientation of D by the vector $(0,0,0,1)$ at the origin is opposite to the orientation given by coordinates x_1, y_1, x_2. Hence,

$$\mathrm{Flux}_D\, \mathbf{v} = -\frac{8\pi\sqrt{2}}{3}.$$

4. Let us consider the complex vector space $\mathbb{C}^2$ with coordinates $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. We can also view $\mathbb{C}^2$ as the real space $\mathbb{R}^4$ with cooedinates (x_1, y_1, x_2, y_2). Denote

$$\alpha := \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2.$$

Take any vector $c = (c_1, c_2) \in \mathbb{C}^2$ of length 1, i.e. $|c_1|^2 + |c_2|^2 = 1$. Denote by Γ_c the circle $\Gamma_c(t) = ce^{2\pi it}$, $t \in [0,1]$. Compute $\int\limits_{\Gamma_c} \alpha$.

We compute the integral directly. Denote $c_1 = a_1 + ib_1, c_2 = a_2 + ib_2$. We have

$$\Gamma_c(t) = ce^{2\pi it} = (c_1 e^{2\pi it}, c_2 e^{2\pi it})$$
$$= (a_1 \cos 2\pi it - b_1 \sin 2\pi it, a_1 \sin 2\pi it + b_1 \cos 2\pi it, a_2 \cos 2\pi it - b_2 \sin 2\pi it, a_2 \sin 2\pi it + b_2 \cos 2\pi it).$$

Hence,

$$\Gamma_c^*\alpha = \pi\big((a_1\cos 2\pi it - b_1\sin 2\pi it)^2 + (a_1\sin 2\pi it + b_1\cos 2\pi it)^2$$
$$+(a_2\cos 2\pi it - b_2\sin 2\pi it)^2 + (a_2\sin 2\pi it + b_2\cos 2\pi it)^2\big)dt = \pi(|c_1|^2 + |c_2|^2)dt = \pi dt.$$

Thus,

$$\int\limits_{\Gamma_c} \alpha = \pi.$$

5. Let us view the space $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) as a complex vector space $\mathbb{C}^2$ with coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$. Consider a surface

$$S = \{(z_1, z_2) \in \mathbb{C}^2;\ z_2 = z_1^2, |z_1| \le 1.\}$$

Compute Area(S).

Let us introduce polar coordinates on complex lines z_1, z_2, i.e. $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$. The the surface S is given by the parameterization

$$(r_1\phi_1) \mapsto F(r_1, \phi_1) = (r_1, \phi_1, r_1^2, 2\phi_1);\ 0 \le r_1 \le 1, 0 \le \phi_1 < 2\pi.$$

The tangent space to the surface is generated by vectors

$$A := \frac{\partial F}{\partial r_1} = \frac{\partial}{\partial r_1} + 2r_1\frac{\partial}{\partial r_2},$$

$$B := \frac{\partial F}{\partial \phi_1} = \frac{\partial}{\partial \phi_1} + 2\frac{\partial}{\partial \phi_2}.$$

The basis $\frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2}, \frac{\partial}{\partial \phi_1}, \frac{\partial}{\partial \phi_1}$ is orthogonal and we have

$$||\frac{\partial}{\partial r_1}|| = ||\frac{\partial}{\partial r_2}|| = 1$$

and

$$||\frac{\partial}{\partial \phi_1}|| = r_1,\ \ ||\frac{\partial}{\partial \phi_2}|| = r_2.$$

Hence,

$$E = \langle A, A \rangle = 1 + 4r_1^2, \;\; G = \langle B, B \rangle = r_1^2 + 4r_2^2 = r_1^2 + 4r_1^4, \;\; F = \langle A, B \rangle = 0.$$

Thus,

$$\sqrt{EG - F^2} = \sqrt{(1 + 4r_1^2)(r_1^2 + 4r_1^4)} = r_1(1 + 4r_1^2).$$

Thus, $\text{Area}(S) = \int\limits_0^{2\pi} \int\limits_0^1 r_1(1 + 4r_1^2) dr_1 d\phi_1 = 2\pi(\frac{1}{2} + 1) = 3\pi.$