Math 52H: Practice problems for the Final Exam

1. Let S^k be the unit sphere in $\mathbb{R}^{k+1}$. Verify whether the map $f : S^k \to S^k$ given by the formula $f(x) = -x$ is homotopic to the identity map.

2. Compute the integral

$$\iint\limits_S (x^2 + y^2) dS,$$

where S is the boundary of the domain $\{\sqrt{x^2 + y^2} \leq z \leq 1\}$.

3. Compute

$$\int\limits_S \frac{dy \wedge dz}{x} + \frac{dz \wedge dx}{y} + \frac{dx \wedge du}{z},$$

where S is the ellipsoid

$$S = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

co-oriented by the outward normal to the domain which it bounds. .

4. Consider a differential form $\omega = \sum\limits_1^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$.

a) Find a vector field $\mathbf{v}$ on $\mathbb{R}^{2n}$ such that

$$d(\mathbf{v} \lrcorner \omega) = \omega.$$

(This problem has infinitely many solutions. Find any of them.)

b) Compute $\mathrm{Flux}_S\,\mathbf{v}$, where S is an ellipsoid

$$\left\{\sum_1^n \frac{x_i^2+y_i^2}{a_i^2}=1\right\}$$

cooriented by the outward normal vector field. Explain why the answer is independent of the choice of $\mathbf{v}$ in Part a).

5. Consider a 4-dimensional submanifold with boundary in $\mathbb{R}^8$:

$$\Gamma = \big\{(x_1,\dots,x_8)\in\mathbb{R}^8;\ x_5=x_1\cos\alpha+x_2\sin\alpha, x_6=-x_1\sin\alpha+x_2\cos\alpha,$$
$$x_7=2x_3-x_4, x_8=-x_3+x_4,\ x_1^2+x_2^2+x_3^2+x_4^4\le 1\big\}.$$

Suppose that Γ is oriented by its parameterization by coordinates (x_1,x_2,x_3,x_4). Compute

$$\int_\Gamma dx_5\wedge dx_6\wedge dx_7\wedge dx_8.$$

6. Consider a vector field

$$\mathbf{v}=\frac{1}{r^3}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right),$$

$r=\sqrt{x^2+y^2+z^2}$ in $\mathbb{R}^3\setminus 0$. Let us denote

$$S:=\left\{(x,y,z)\in\mathbb{R}^3;\ z=e^{x^2+y^2-\frac{1}{2}},\ x^2+y^2+z^2\le\frac{3}{2}\right\}$$

and co-orient this surface by a normal vector field which is equal to $(0,0,1)$ at the point $(0,0,\frac{1}{\sqrt{e}})\in S$. Compute $\mathrm{Flux}_S\,\mathbf{v}$.

7. Suppose that a vector field $\mathbf{v}$ in $\mathbb{R}^3$ with coordinate functions (P,Q,R) satisfies $\mathrm{curl}\,\mathbf{v}=0$. Find an explicit expression for a function F such that $\mathbf{v}=\nabla F$.

8. Let C be the intersection of the sphere $S=\{x^2+y^2+z^2=1\}$ and the plane $P=\{x+y+z=0\}$. We orient C counter-clockwise when looking from the point $(0,0,100)$.

Compute $\int_C z^3dx$.

9. Let M be an oriented closed n-dimensional manifold, and ω be a differential $(n-1)$-form on M. Prove that there exists a point $a\in M$ such that $(d\omega)_a=0$.

The actual final exam will consist of 5 problems.