Math 52H: Solutions to practice problems for the Final Exam

1. Let S^k be the unit sphere in $\mathbb{R}^{k+1}$. Verify whether the map $f : S^k \to S^k$ given by the formula $f(x) = -x$ is homotopic to the identity map.

Suppose that k is odd, $k = 2n-1$. Then S^k can be viewed as the unit sphere in $\mathbb{C}^n$. Consider a family of maps $\theta_t : \mathbb{C}^n \to \mathbb{C}^n$ given by the formula

$$\theta_t(z) = e^{\pi i t}z = (e^{\pi i t}z_1, \dots, e^{\pi i t}z_n),\ z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

The the restriction of this homotopy to the unit sphere is a homotopy between the identity map and the antipodal map. Suppose next that k is even, $k = 2n$. Then S^k can be viewed as the unit sphere in R^{2n+1}. The volume form on S^k is the restriction to S^k of the form

$$\alpha = \sum_1^{2n+1} (-1)^{j-1} x_j dx_1 \wedge \dots \overset{j}{\check{}} \dots dx_{2n+1}.$$

If $f(x) = -x$ then $f^*\alpha = (-1)^{2n+1}\alpha = -\alpha$. Hence $\int_{S^k} f^*\alpha = -\int_{S^k} \alpha > 0$, and therefore f is not homotopic to the identity map.

2. Compute the integral

$$\iint\limits_S (x^2 + y^2) dS,$$

where S is the boundary of the domain $\{\sqrt{x^2+y^2} \le z \le 1\}$.

The surface S is the union of the surface $P = \{z = \sqrt{x^2+y^2}; x^2+y^2 \le 1\}$ and the disc $\Delta = \{z = 1; x^2+y^2 \le 1\}$. Let us coordinatize both surfaces via the projection to the plane (x, y). Then the area form on Δ is just $\sigma_\Delta = dx \wedge dy$ and to compute σ_P we use the parametrization $\Phi(x, y) = x, y, r = \sqrt{x^2+y^2}$. Then $\Phi_x = (1, 0, \frac{x}{r})$, $\Phi_y = (0, 1, \frac{y}{r})$. Thus $E = 1 + \frac{x^2}{r^2}, G = 1 + \frac{y^2}{r^2}$ and $F = \frac{xy}{r^2}$. Hence

$$EG - F^2 = (1 + \frac{x^2}{r^2})(1 + \frac{y^2}{r^2}) - \frac{x^2y^2}{r^4} = 2.$$

Thus $\sigma_P = \sqrt{2}dx \wedge dy$. Denote $D = \{x^2+y^2 \le 1\} \subset \mathbb{R}^2$. We need to compute two integrals

$$I_1 = \int\limits_D (x^2+y^2)dx \wedge dy = \int\limits_0^{2\pi}\int\limits_0^1 r^3 dr d\phi = \frac{\pi}{2}$$

and

$$I_2 = \sqrt{2}\int\limits_D (x^2+y^2)dxdy = \frac{\pi}{\sqrt{2}}.$$

Hence the answer is $\frac{\pi(1+\sqrt{2})}{2}$.

3. Compute

$$\int\limits_S \frac{dy \wedge dz}{x} + \frac{dz \wedge dx}{y} + \frac{dx \wedge dy}{z},$$

where S is the ellipsoid

$$S = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

co-oriented by the outward normal to the domain which it bounds.

Denote $\eta := \frac{dy\wedge dz}{x} + \frac{dz\wedge dx}{y} + \frac{dx\wedge dy}{z}$. Let us rescale variables:

$$X = \frac{x}{a}, Y = \frac{y}{b}, Z = \frac{z}{c}.$$

The S becomes the unit sphere $X^2+Y^2+Z^2=1$ and the form η can be written as $\frac{A}{X}dY\wedge dZ+\frac{B}{Y}dZ\wedge dX+\frac{C}{Z}dX\wedge dY$, where

$$A=\frac{bc}{a},B=\frac{ca}{b},C=\frac{ab}{c}.$$

We note that $\int_S \eta = \mathrm{Flux}_S\,\mathbf{v}$, where $\mathbf{v}$ is the vector field with coordinate functions $\frac{A}{X},\frac{B}{Y},\frac{C}{Z}$. Hence,

$$\int\limits_S \eta=\int\limits_S (A+B+C)dS=(A+B+C)\mathrm{Area}(S)=4\pi\frac{(ab)^2+(bc)^2+(ca)^2}{abc}.$$

4. Consider a differential form $\omega=\sum\limits_1^n dx_i\wedge dy_i$ on $\mathbb{R}^{2n}$.

a) Find a vector field $\mathbf{v}$ on $\mathbb{R}^{2n}$ such that

$$d(\mathbf{v}\,\lrcorner\,\omega)=\omega.$$

(This problem has infinitely many solutions. Find any of them.)

b) Compute $\mathrm{Flux}_S\,\mathbf{v}$, where S is an ellipsoid

$$\left\{\sum_1^n \frac{x_i^2+y_i^2}{a_i^2}=1\right\}$$

cooriented by the outward normal vector field. Explain why the answer is independent of the choice of $\mathbf{v}$ in Part a).

a) One of the solutions is $\mathbf{v}=\sum\limits_1^n y_i\frac{\partial}{\partial y_i}$. Indeed, $\mathbf{v}\,\lrcorner\,\omega=-\sum\limits_1^n y_i dx_i$ and $d(\mathbf{v}\,\lrcorner\,\omega)=\omega$. Recall that the volume form $\Omega=dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n$ is equal to$\frac{1}{n!}\omega^n$. Hence, we have

$$\mathbf{v}\,\lrcorner\,\Omega=\frac{1}{n!}\mathbf{v}\,\lrcorner\,\omega^n=\frac{1}{(n-1)!}\mathbf{v}\,\lrcorner\,\omega\wedge\omega^{n-1}.$$

In particular,

$$d(\mathbf{v}\,\lrcorner\,\Omega)=\frac{1}{(n-1)!}\omega^n=n\Omega.$$

b) By Stokes theorem we have

$$\mathrm{Flux}_S\,\mathbf{v} = \int\limits_S \mathbf{v} \lrcorner \Omega = \int\limits_U d(\mathbf{v} \lrcorner \Omega) = n \int\limits_U \Omega = n\mathrm{Vol}U,$$

where we denote by U the solid ellipsoid bounded by S.

Note, that $\mathrm{Vol}U = a_1^2 \dots a_n^2 \mathrm{Vol}B_1$ where B_1 is the unit ball in $\mathbb{R}^{2n}$. We recall that $\mathrm{Vol}B_1 = \frac{\pi^n}{n!}$

5. Consider a 4-dimensional submanifold with boundary in $\mathbb{R}^8$:

$$\Gamma = \big\{(x_1,\dots,x_8) \in \mathbb{R}^8;\ x_5 = x_1\cos\alpha + x_2\sin\alpha, x_6 = -x_1\sin\alpha + x_2\cos\alpha,$$
$$x_7 = 2x_3 - x_4, x_8 = -x_3 + x_4,\ x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 1\big\}.$$

Suppose that Γ is oriented by its parameterization by coordinates (x_1, x_2, x_3, x_4). Compute

$$\int\limits_\Gamma dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8.$$

Parameterizing Γ by coordinates x_1, x_2, x_3, x_4 and expressing $dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8$ in these coordinates we get

$$dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8 = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

Hence, the integral is equal to the volume of the init 4-ball, i.e. $\frac{\pi^2}{2}$.

6. Consider a vector field

$$\mathbf{v} = \frac{1}{r^3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right),$$

$r = \sqrt{x^2+y^2+z^2}$ in $\mathbb{R}^3 \setminus 0$. Let us denote

$$S := \left\{(x,y,z) \in \mathbb{R}^3;\ z = e^{x^2+y^2-\frac{1}{2}},\ x^2+y^2+z^2 \le \frac{3}{2}\right\}$$

and co-orient this surface by a normal vector field which is equal to $(0,0,1)$ at the point $(0,0,\frac{1}{\sqrt{e}}) \in S$. Compute $\mathrm{Flux}_S\,\mathbf{v}$.

The equation $e^{2r^2-1} + r^2 = \frac{3}{2}$ has a solution $r^2 = \frac{1}{2}$, and hence the surface S is bounded by the circle $\Gamma = \{x^2 + y^2 = \frac{1}{2}, z = 1\}$. The normal component of the vector field $\mathbf{v}$ to the unit sphere has the length $\sqrt{3}2$ (equal to the radius of the sphere). Hence, the question amonts to a computation of the area of the spherical cap bounded by Γ. Let us compute the area form. The surface given by a parameterizing map $\Phi(x, y) = (x, y, S = \sqrt{\frac{3}{2} - x^2 - y^2}$. We have $\Phi_x = (1, 0, -\frac{x}{S}), \Phi_y = (0, 1, \frac{y}{S})$. Thus,

$$E = 1 + \frac{x^2}{S^2},\ G = 1 + \frac{y^2}{S^2},\ F = \frac{xy}{S^2}.$$

Hence,

$$EG - F^2 = 1 + \frac{x^2}{S^2} + \frac{y^2}{S^2} = \frac{3}{3 - 2x^2 - 2y^2},$$

and

$$\text{Area}(S) = 3 \int\limits_{x^2+y^2 \leq 12} \frac{dxdy}{\sqrt{3 - 2x^2 - 2y^2}} = 3 \int\limits_0^{2\pi} \int\limits_0^{\frac{1}{\sqrt{2}}} \frac{rdrd\phi}{\sqrt{3 - 2r^2}} = 3\pi \int\limits_0^{\frac{1}{2}} \frac{du}{\sqrt{3 - 2u}} = 3\pi(\sqrt{3} - \sqrt{2}).$$

Finally to get the flux we need to multiply the area by $\sqrt{32}$.

7. Suppose that a vector field $\mathbf{v}$ in $\mathbb{R}^3$ with coordinate functions (P, Q, R) satisfies $\operatorname{curl} \mathbf{v} = 0$. Find an explicit expression for a function F such that $\mathbf{v} = \nabla F$.

The equation $\operatorname{curl} \mathbf{v} = 0$ is equivalent to $d\alpha = 0$, where We have $\alpha := \mathcal{D}(\mathbf{v}) = Pdx + Qdy + Rdz$. In $\mathbb{R}^3$ the closed form α is exact and its primitive F (i.e. $dF = \alpha$) can be computed by the formula

$$F(u) = \int\limits_0^1 (xP(tu) + yP(tu) + zP(tu))\, dt.$$

where $u = (x, y, z)$. The equation $dF = \alpha$ is equivalent to $\nabla F = \mathbf{v}$.

8. Let C be the intersection of the sphere $S = \{x^2 + y^2 + z^2 = 1\}$ and the plane $P = \{x + y + z = 0\}$. We orient C counter-clockwise when looking from the point $(0, 0, 100)$. Compute $\int\limits_C z^3 dx$.

Let us use Stokes' theorem applied to the disc Δ bounded by the circle C in the plane $\{x+y+z=0\}$. The corresponding orientation of Δ coincides with its orientation by coordinates (x,y) via the orthogonal projection. We have

$$I := \int\limits_C z^3 dx = \int\limits_\Delta 3z^2 dz \wedge dx.$$

Expressing in coordinates x,y we get $z = -(x+y)$ and

$$3z^2 dz \wedge dx = 3(x+y)^2 dx \wedge dy.$$

Disc D projects to the plane (x,y) as a (solid) ellipse $E = \{x^2+y^2+(x+y)^2 \leq 1\}$. By rotating the axes by $\pi/4$, $u = \frac{\sqrt{2}}{2}(x-y), v = \frac{\sqrt{2}}{2}(x+y)$ we can rewrite the equation of the solid ellipse as $u^2+3v^2 \leq 1$. Note that in the new coordinates $dx \wedge dy = du \wedge dv$ and $(x+y)^2 = 2v^2$.

$$I = \int\limits_\Delta 3z^2 dx \wedge dy = 3\int\limits_E (x+y)^2 dx \wedge dy = 6 \iint\limits_{u^2+3v^2 \leq 1} v^2 du dv = 6\int\limits_{-1}^{1}\int\limits_{-S}^{S} v^2 dv du,$$

where we denoted $S := \frac{1}{\sqrt{3}}\sqrt{1-u^2}$. We further have

$$I := 6\int\limits_{-1}^{1}\int\limits_{-S}^{S} v^2 dv du = 4\int\limits_{-1}^{1} S^3 du = \frac{4}{3\sqrt{3}}\int\limits_{-1}^{1}(1-u^2)^{\frac{3}{2}} du.$$

Substituting $u = \sin t$ we get

$$I = \frac{4}{3\sqrt{3}}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t dt = \frac{1}{6\sqrt{3}}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(3+\cos 2t + 4\cos 4t)dt = \frac{\pi}{2\sqrt{3}}.$$

9. Let M be an oriented closed n-dimensional manifold, and ω be a differential $(n-1)$-form on M. Prove that there exists a point $a \in M$ such that $(d\omega)_a = 0$.

By Stokes theorem we have $\int\limits_M \omega = 0$. The n-form ω is proportional to the volume for σ_M, $\omega = f\sigma_M$ and we have $\int\limits_M \omega = \int f dV$. Hence, the function f should change sign and thus by continuity at some point $f(a) = 0$, and hence $\omega_a = f(a)(\sigma_M) = 0$.