Math 52H: Practice problems for the midterm

1 Consider a Euclidean space $V = \mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$ and the standard dot-product. The space $\Lambda^k(V^*)$ of all exterior k-forms is also a Euclidean space with the scalar product of defined by the formula

$$\langle\langle \alpha, \beta \rangle\rangle \;=\; \star^{-1}(\alpha \wedge \star\beta).$$

Consider a linear operator $\Omega : \Lambda^k(V^*) \to \Lambda^{k+2}(V^*)$ defined by the formula $\Omega(\alpha) = \alpha \wedge \omega$, where $\omega = \sum\limits_1^n x_i \wedge y_1$. Find the adjoint linear operator $\Omega^\star$, i.e. the operator

$$\Omega^\star : \Lambda^{k+2}(V^*) \to \Lambda^k(V^*)$$

such that

$$\langle\langle \Omega(\alpha), \beta \rangle\rangle = \langle\langle \alpha, \Omega^\star(\beta) \rangle\rangle$$

for any forms $\alpha \in \Lambda^k(V^*), \beta \in \Lambda^{k+2}(V^*)$.

2. Consider two differential 1-forms in $\mathbb{R}^3$:

$$\alpha = dx + ydz \quad \text{and} \quad \beta = xdy\,.$$

Prove that there is no map $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f^*(\beta) = \alpha$.

3. The cylindrical coordinates

$$r \in [0, \infty),\ \varphi \in [0, 2\pi),\ z \in \mathbb{R}\,,$$

are introduced in $\mathbb{R}^3$ by the formulas

$$x = r\cos\varphi, y = r\sin\varphi\,,$$

where (x, y, z) are Cartesian coordinates. Consider a differential 1-form

$$\alpha = \cos r dz + \frac{r\sin r}{\pi} d\varphi\,.$$

Suppose that a curve $\Gamma \subset \mathbb{R}^3$ is given parametrically by a map

$$[0,\pi] \xrightarrow{\Gamma} \mathbb{R}^3,$$

$$\Gamma(t) = (r = \frac{\pi}{4},\, z = h(t),\, \varphi = 2t),\ t \in [0,\pi]\,.$$

Find the function h such that $\Gamma^*\alpha = 0$ and $h(0) = 1$.

4. Consider a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$. Let S_f be a surface in $\mathbb{R}^4$ given by equations

$$x_3 = \frac{\partial f}{\partial x_1}(x_1, x_2), \quad x_4 = \frac{\partial f}{\partial x_2}(x_1, x_2) \tag{1}$$

Suppose that this system of equations can is solved with respect to the coordinates x_2 and x_4, i.e. there exist smooth functions $x_2 = g(x_1, x_3)$ and $x_4 = h(x_1, x_3)$ such that

$$\begin{aligned} x_3 &\equiv \frac{\partial f}{\partial x_1}(x_1, g(x_1, x_3)), \\ h(x_1, x_3) &\equiv \frac{\partial f}{\partial x_2}(x_1, g(x_1, x_3)). \end{aligned} \tag{2}$$

Prove that the Jacobian of the map $(h, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is equal to -1, i.e. that

$$\begin{vmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial g}{\partial x_3} \end{vmatrix} = -1.$$

Hint: Consider the pull-back of the form $\omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$ by a map $\mathbb{R}^2 \to S_f \subset \mathbb{R}^4$ given by the formulas

$$(x_1, x_3) \mapsto (x_1, g(x_1, x_3), x_3, h(x_1, x_3))\,.$$

5. Consider a smooth differential k-form

$$\alpha = \sum_{1 \le i_1 < \dots < i_k \le n} f_{i_1 \dots i_k}(x) dx_{i_1} \wedge \dots dx_{i_k}$$

in $\mathbb{R}^n$ such that $f_{i_1 \dots i_k}(0) = 0$ (i.e. all coefficients of the form α are equal to 0 at the origin). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ denote the dilatation $x \mapsto 2x$. Suppose that $F^*\alpha = \alpha$. Prove that $\alpha \equiv 0$.

6. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, consider a map $F : \mathbb{R}^n \to \mathbb{R}^{2n+1}$ defined by the formula

$$F(x_1, \dots, x_n) = \left(x_1, \dots, x_n, \frac{\partial f}{\partial x_1}(x_1, \dots, x_n), \dots, \frac{\partial f}{\partial x_n}(x_1, \dots, x_n), f(x_1, \dots, x_n) \right) .$$

Compute $F^*(\alpha)$, where

$$\alpha = dx_{2n+1} - \sum_{i=1}^{n} x_{i+n} dx_i \, .$$

The actual midterm will consist of four problems.