Math 52H Homework 9 Solutions

March 8, 2013

1. By the divergence Theorem:

$$\iint_S \langle \mathbf{n}, \mathbf{v} \rangle = \text{Flux}_S(\mathbf{v}) = \int_{\mathbf{U}} \text{div}(\mathbf{v}) = \mathbf{0}.$$

Here U is the domain bounded by S.

2. a) We have $\frac{\partial u}{\partial \mathbf{n}} = \langle \nabla u, \mathbf{n} \rangle$. We also note that $\text{div}\nabla u = \Delta u$. Hence, harmonicity of u is equivalent to the fact that $\text{div}\nabla u = 0$. Hence, if u is harmonic then the divergence theorem implies that $\oint_\Gamma \frac{\partial u}{\partial \mathbf{n}} ds = 0$, Conversely, applying

$$\oint_\Gamma \frac{\partial u}{\partial \mathbf{n}} ds = 0, \tag{1}$$

to circles $S_\epsilon(a)$ of radius ϵ centered at a point $a \in U$

$$\Delta u(a) = \text{div}\nabla u(a) = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \oint_{S_\epsilon(a)} \frac{\partial u}{\partial \mathbf{n}} ds = 0.$$

b) We have

$$\oint_\Gamma u \frac{\partial u}{\partial \mathbf{n}} ds = \text{Flux}_\Gamma(u \nabla u).$$

Furthermore,

$$\text{div}(u\nabla u) = u\Delta u + \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2.$$

Hence, the required formula is just the divergence theorem for $u\nabla u$.

c) This is very similar to the previous part b): it is enough to apply the divergence theorem for the vector field $v\nabla u - u\nabla v$.

3. a. Write $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$. Assume that there is a primitive F. If we integrate α_1 with respect to x, we get

$$F = x^3/3 - x^2/2 + xy + xz - xyz + f(y,z),$$

for some function $f(y,z)$ depending only on y and z. Differentiate this w.r.t. y and set it equal to α_2 to get

$$x - xz + \frac{\partial f}{\partial y} = y^2 + x - y + z - xz,$$

which is equivalent to $f = y^3/3 - y^2/2 + yz + g(z)$ for some function g depending only on z. Now differentiate the expression we have for F w.r.t. z and set equal to α_3 to get

$$x - xy + y + g'(z) = z^2 + x + y - z - xy,$$

whence $g(z) = z^3/3 - z^2/2 + C$ for some constant C. We thus have

$$F(x,y,z) = \frac{x^3 + y^3 + z^3}{3} - \frac{x^2 + y^2 + z^2}{2} + xy + xz + yz - xyz + C.$$

Remark: That we were able to find a solution to the equations above implies that α is exact with F as its primitive. In fact, the equations we solved above imply that $dF = \alpha$. Also, note that finding a primitive for an exact 1-form is the same as finding a potential function for a vector field. Finally, we remark here that one can guess the form of F very easily by symmetry.

4. If we let r to be the distance from the origin in $\mathbb{R}$, namely $r = \sqrt{x^2 + y^2 + z^2}$ then we have by visual inspection that $\mathbf{v} = r^2 \mathbf{r}$ and therefore $\mathbf{v} = \nabla f$ for $f = r^4/4$. As a result the work we need to compute is given by difference of the values of the potential at the end-points of the path:

$$\int_\Gamma \mathbf{v} \cdot \mathbf{t} = \int_\Gamma \nabla f \cdot \mathbf{t} = \int_\Gamma df = f(B) - f(A).$$

Therefore

$$\mathrm{Work}_\Gamma(\mathbf{v}) = \frac{1}{4}\left[\|b\|^4 - \|a\|^4\right].$$