Math 52H Homework 8 Solutions

March 11, 2013

1. a. Write $\alpha = \frac{xdx+ydy+zdz}{(x^2+y^2+z^2)^{3/2}}$ and let $G : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be defined by $G(x,y,z) = \frac{-3}{4}(x^2+y^2+z^2)^{\frac{-1}{2}}$. Then

$$dG = \frac{-3}{4}(x^2+y^2+z^2)^{\frac{-3}{2}}(\frac{-2}{3})(2xdx+2ydy+2zdz) = \alpha$$

so α is exact on $\mathbb{R}^3 \setminus \{0\}$, which contains the curve γ. It follows that

$$\int_\gamma \alpha = G(\gamma(1)) - G(\gamma(0)) = \frac{3}{4}(\frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{3}}).$$

b. Write $\alpha = \frac{xdy-ydx}{x^2+y^2}$. On the set $\{(x,y) : x > 0\}$ we have $\alpha = d\arctan(\frac{y}{x})$, so γ is exact on this domain. Since $\gamma_x(t) = 1 + \frac{1}{2}\sin\frac{t^2}{\pi} > 0$ for all t, the curve γ is contained in $\{(x,y) : x > 0\}$. Hence

$$\int_\gamma \alpha = \arctan(\frac{\gamma_y(\pi)}{\gamma_x(\pi)}) - \arctan(\frac{\gamma_y(0)}{\gamma_x(0)}) = 0 - \frac{\pi}{4} = \frac{-\pi}{4}.$$

2. Recall $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$. Hence, since ∂A is empty,

$$0 = \int_{\partial A} \omega \wedge \eta = \int_A d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

It follows

$$\int_A \omega \wedge d\eta = (-1)^{k+1} \int_A d\omega \wedge \eta = (-1)^{k+1+(k+1)l} \int_A \eta \wedge d\omega.$$

The constant C may thus be taken to be $(-1)^{(k+1)(l+1)}$.

3. Let $\gamma : [0,1] \to \mathbb{R}^2$ be a C^1- parametrization of the curve Γ. Then, by definition of integral of a 1-form along a curve we have

$$\left| \int_\Gamma P dx + Q dy \right| = \left| \int_0^1 \left[P(\gamma(t))\gamma_1'(t) + Q(\gamma(t))\gamma_2'(t) \right] dt \right|$$

and then by the Cauchy-Schwartz inequality for vectors in $\mathbb{R}^2$ we get

$$\left| \int_\Gamma P dx + Q dy \right| \leq \int_0^1 \|(P,Q)\| \, \|\gamma'(t)\| \; dt \leq \max_\Gamma \sqrt{P^2+Q^2} \int_0^1 \|\gamma'(t)\| \; dt \leq L \max_\Gamma \sqrt{P^2+Q^2}.$$

4. If A is the area we need to compute, we have by Stokes' Theorem (in the Green-Gauss form) that $A = \int_C -y dx$ where C is the astroid, parametrised by $x(t) = a\cos^3 t, \; y(t) = b \sin^3 t$ with $a, b \in \mathbb{R}_{>0}$ and $0 \leq t \leq 2\pi$. Hence

$$A = \int_C -y dx = 3ab \int_0^{2\pi} \left[\sin^4(t) - \sin^6(t) \right] \, dt.$$

By Wallis formula (that can be proved inductively, using integration by parts)

$$\int_0^{\pi/2} \sin^{2n}(t) \, dt = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}$$

where $k!!$ stands for the semi-factorial of k, the product of all positive integers not bigger than k and having the same parity. As a result one gets

$$A = \frac{3}{8}\pi ab.$$

5. This follows immediately from the solution of Problem 3 above, since clearly

$$P = \frac{y}{(x^2+xy+y^2)^2}, \;\; Q = \frac{-x}{(x^2+xy+y^2)^2}$$

implies that on S_R one has

$$\max_{S_R} \sqrt{P^2+Q^2} \leq \frac{10}{R^3}$$

and since $L(S_R) = 2\pi R$ the fact that $I(R) \to 0$ for $R \to \infty$ follows at once.

6. Let us work in standard spherical coordinates in $\mathbb{R}^3$ so that the region in question Σ is described by $\theta \in (0, 2\pi)$ and $\phi \in (0, \phi_0)$. Let us notice explicitly that $\phi_0 < \frac{\pi}{2}$. If F is the parametrization of Σ in spherical coordinates we get that

$$\left\| \frac{\partial F}{\partial \phi} \wedge \frac{\partial F}{\partial \theta} \right\| = R^2 \sin \phi$$

hence

$$\text{Area}\,(\Sigma) = R^2 \int_0^{2\pi} \int_0^{\phi_0} \sin \phi \, d\phi d\theta = 2\pi R^2 \left(1 - \cos(\phi_0)\right).$$

The problem is then solved by picking $R \simeq 6.3 \times 10^6 m$ and $\phi \simeq 67°$.

7. Let D be the domain inside the ellipse C. Then by Stokes' Theorem

$$\int_C \alpha = \int_D d\alpha$$

where $\alpha = (y + z)\, dx + (z + x)\, dy + (x + y)\, dz$. It is checked at once that $d\alpha = 0$ and therefore the integral to be computed is zero as well.