Math 52H Homework 7 Solutions

March 1, 2013

1. Let $S^n \hookrightarrow \mathbb{R}^{n+1}$ be the standard unit sphere and let us work with the following notation: $x = (x_1, \ldots, x_{n+1})$ and $x' = (x_1, \ldots, x_n)$. Finally, let (U_+, p_+) and (U_-, p_-) the two stereographic charts. Hence the transition maps we need to compute are given by $h_{-+} = p_- \circ p_+^{-1}$ and $h_{+-} = p_+ \circ p_-^{-1}$. We have already seen in class that

$$p_+(x) = \frac{x'}{1 - x_{n+1}}, \; p_-(x) = \frac{x'}{1 + x_{n+1}}$$

and we need to compute their inverse maps. For instance, let's see p_+^{-1}. Arguing geometrically, let us pick the line $(1-t)(0, 0, \ldots, 0, -1) + t(x', 0)$ and determine t by requiring the corresponding point to belong to S^n: this gives $t = 2/(1 + |x'|^2)$ and therefore

$$p_+^{-1} = \left(\frac{2x'}{1 + |x'|^2}, \frac{1 - |x'|^2}{1 + |x'|^2} \right).$$

As a result, one finds that $h_{-+}(x') = \frac{x'}{|x'|^2}$ and the same is true for the other transition map. Namely, these are the standard spherical inversions with respect to $S^{n-1} \hookrightarrow \mathbb{R}^n$.

2.

a) Let us recall that $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \approx$ where we declare $x \approx y$ iff there is $\lambda \in \mathbb{R}_{\neq 0}$ such that $y = \lambda x$. Let us call $p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ the projection map. We can define $n+1$ charts of *homogeneous coordinates* by defining (on $U_j = p\left(\tilde{U}_j\right)$ where $\tilde{U}_j = \{x \in \mathbb{R}^{n+1} \setminus \{0\} \mid x_j \neq 0\}$, $j = 1, 2, \ldots, n+1$)

$$\pi_j(x) = \left[\frac{x_1}{x_j}; \frac{x_2}{x_j}; \ldots; \frac{x_{n+1}}{x_j} \right].$$

Clearly $\bigcup_{j=1}^{n+1} \tilde{U}_j = \mathbb{R}^{n+1} \setminus \{0\}$ so that $\bigcup_{j=1}^{n+1} U_j = \mathbb{R}P^n$, the inverse map π_i^{-1} is given by

$$\pi_i^{-1}([y_1; y_2; \dots ; y_n]) = (y_1, y_2, \dots , y_{i-1}, 1, y_i, \dots , y_n)$$

and the transition maps are given, for $i \neq j$ by

$$h_{ij}(x) = \pi_j \circ \pi_i^{-1}(y) = \left[\frac{y_1}{y_j}; \frac{y_2}{y_j}; \dots ; \frac{y_{i-1}}{y_j}; \frac{1}{y_j}; \frac{y_i}{y_j}; \dots ; \frac{y_n}{y_j}\right]$$

so these are *algebraic, in fact rational, functions*.

b) The key point for this part of the exercise is to show that in fact $SO(3)$ is diffeomorphic to the 3-dimensional real projective space $\mathbb{R}P^3$ and thus it inherits the corresponding atlas (described above) through this identification. This diffeomorphism is well-known and can be found in several books and in other resources on-line.

A different approach (indeed a simpler one, in some sense) is to show that 1) $SO(3)$ is in bijection with the unit tangent bundle of S^2, 2) the unit tangent bundle of S^2 is a smooth submanifold of TS^2 and hence to obtain an atlas by restriction of the coordinate charts of TS^2 itself. For the first part, it's enough to observe that the columns of a matrix in $SO(3)$ form a (positive) orthonormal basis of $\mathbb{R}^3$. Hence, given $A \in SO(3)$ we first associate to it a point of $S^2 \hookrightarrow \mathbb{R}^3$ by considering the first column and then we use the second column to identify a vector in the unit tangent bundle of S^2 at the corresponding point. The third column vector of A is uniquely determined by the first two and therefore is irrelevant in describing this bijection. For the second part, I will explicitly refer to the solution of c) below for the notation etc... There, I have shown that TS^2 has an atlas made of two charts $\{(V_+, F_+), (V_-, F_-)\}$ where p_+, p_- are the maps defined in problem 1. Given one of these charts, say V_+ let x_1, x_2, y_1, y_2 the corresponding coordinates. Let us define $w \in T_{p_+^{-1}(x_1,x_2)}S^2 \hookrightarrow \mathbb{R}^3$ by $z = dp_+^{-1}(x_1, x_2)[y_1, y_2]$. So z is a well-defined and smooth function of x_1, x_2, y_1, y_2. Then

$$UT(S^2) \cap V_+ = \left\{(x_1, x_2, y_1, y_2) \in V_+ \simeq \mathbb{R}^4 |\, |z|^2 = 1\right\}$$

and so by the implicit function theorem this implies that $UT(S^2)$ is a smooth submanifold of TS^2. Hence, by general arguments, it inherits a smooth atlas (possibly with more than two charts) and the transition maps are smooth as well.

c) We have an atlas of two charts which is *induced* by the corresponding stereographic atlas on S^2 (see the first exercise). So I will denote such atlas by $\{(V_+, F_+), (V_-, F_-)\}$ where for instance $V_+ = p_+^{-1}TS^2$ so it is the set of applied vectors whose base-point is NOT the south pole of S^2 (here p_+ is the map defined in problem 1). On V_+ we have coordinates (x_1, x_2, y_1, y_2) where y_1, y_2 are determined by the requirement that for a given vector field Θ we have

$$dp_+\Theta = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}.$$

The very same definition applies to V_-. Now we need to determine the transition maps, say $k_{-+} : \mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \hookleftarrow$. Clearly we have $k_{-+}(x_1, x_2, y_1, y_2) = (h_{-+}(x_1, x_2), l_{-+}(x_1, x_2, y_1, y_2))$ with $l_{-+}(x_1, x_2, y_1, y_2)$ to be determined. To this aim we proceed as follows: let Θ be a vector field on S^2 and let

$$dp_+\Theta = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}$$

we need to determine z_1, z_2 as functions of x_1, x_2, y_1, y_2 so that

$$dp_-\Theta = z_1 \frac{\partial}{\partial u_1} + z_2 \frac{\partial}{\partial u_2}$$

where we have set u_1, u_2 the local coordinates associated to the projection p_- to avoid confusion with x_1, x_2 (that are induced by the map p_+ instead). Since $u_j(x) = \frac{x_j}{|x|^2}$ for $j = 1, 2$ we get

$$\frac{\partial u_1}{\partial x_1} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}, \quad \frac{\partial u_1}{\partial x_2} = \frac{-2x_1 x_2}{(x_1^2 + x_2^2)^2}$$

and symmetrically

$$\frac{\partial u_2}{\partial x_1} = \frac{-2x_1 x_2}{(x_1^2 + x_2^2)^2}, \quad \frac{\partial u_2}{\partial x_2} = \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}$$

and finally we use them in applying the chain rule:

$$\frac{\partial}{\partial x_1} = \frac{\partial u_1}{\partial x_1} \frac{\partial}{\partial u_1} + \frac{\partial u_2}{\partial x_1} \frac{\partial}{\partial u_2}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial u_1}{\partial x_2} \frac{\partial}{\partial u_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial}{\partial u_2}$$

so that we first obtain

$$l_{ij}(x_1, x_2, y_1, y_2) = \left(y_1 \frac{\partial u_1}{\partial x_1} + y_2 \frac{\partial u_1}{\partial x_2}, y_1 \frac{\partial u_2}{\partial x_1} + y_2 \frac{\partial u_2}{\partial x_2} \right)$$

and finally

$$l_{ij}(x_1, x_2, y_1, y_2) = \left(y_1 \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2} - 2y_2 \frac{x_1 x_2}{(x_1^2 + x_2^2)^2}, -2y_1 \frac{x_1 x_2}{(x_1^2 + x_2^2)^2} + y_2 \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2} \right).$$

3. Write F for the parametrization $F(x_1, ..., x_n) = \left(x_1, ..., x_n, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right)$ and $\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$. By previous homework, wedge and pullback commute. Then

$$\int_{L_f} \omega^{\wedge k} = \int_{F^{-1}(L_f)} F^*(\omega^{\wedge k}) = \int_{F^{-1}(L_f)} (F^*\omega)^{\wedge k}.$$

But we calculate

$$F^*\omega = \sum_{i=1}^{n} dx_i \wedge \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \right) = \sum_{i \neq j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j = \sum_{i<j} \frac{\partial^2 f}{\partial x_i \partial x_j} (dx_i \wedge dx_j + dx_j \wedge dx_i) = 0.$$

Hence the integral vanishes.

4. Let V denote the submanifold given by $x^2 + y^2 + z^2 \leq 1$ with $z \geq 0$, and let D denote the disc of radius 1 centered at the origin and lying on the xy plane oriented with upwards pointing normal. Then $\partial V = S \cup D$. Let $\omega = dx \wedge dy + z dz \wedge dx$. Note that $d\omega = 0$, and so by Stokes Theorem

$$0 = \int_V d\omega = \int_S \omega - \int_D \omega.$$

On D, $z = 0$ so $\int_S \omega = \int_D dx \wedge dy = \pi$, the latter being just the area of the disc D.

To compute directly, parametrize S by $P(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$ for $(x, y) \in D$. Note that the tangent plane is spanned by $T_x = (1, 0, \text{something})$ and $T_y = (1, 0, \text{something})$ so $T_x \times T_y$ always has positive z component.

Now

$$dz = \frac{2}{\sqrt{1 - x^2 - y^2}}(-2x dx - 2y dy),$$

so

$$z dz \wedge dx = z \frac{y}{\sqrt{1 - x^2 - y^2}} dx \wedge dy = y dx \wedge dy.$$

Thus $\int_S \omega = \int_D (1 + y) dx \wedge dy = \pi$ since the first term gives the area of the the disc, and the second is odd in y while D is symmetric in y.