Math 52H Homework 6 Solutions

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1. The equation $(r-2)^2+z^2=1$ describes a torus in $\mathbb{R}^3$ of small radius 1 centered around a circle of radius 2 in the x-y plane. $T \subset \mathbb{R}^4$ can be imagined as the direct product of two unit circles in two orthogonally intersecting copies of $\mathbb{R}^2$ in $\mathbb{R}^4$ given by (x_1, x_2) and (x_3, x_4), respectively. The first pair (x_1, x_2) should describe which ϕ-slice of the torus we are on, and the second pair (x_3, x_4) should describe where on that ϕ-slice we are. With this geometric intuition in mind, we define the diffeomorphism:

$$\begin{aligned} x_1 &= \cos\phi \\ x_2 &= \sin\phi \\ x_3 &= r-2 \\ x_4 &= z. \end{aligned}$$

First, observe that this is clearly a C^∞ map. So it suffices to check that it is a bijection.

Surjective: As ϕ varies, clearly we hit all (x_1, x_2) on the unit circle. As the equations $x_3 = r-2$ and $x_4 = z$ are linear, and the constraint equations $x_3^2+x_4^2 = 1$ and $(r-2)^2+z^2 = 1$ are identical, we conclude that we hit all possible admissible choices of x_3 and x_4 as well, as r and z vary. ϕ is independent of r, z, so in fact, the map is surjective.

Injective: We must restrict ourselves to $\phi \in [-\pi, \pi)$ to make the map to (x_1, x_2) injective. We don't lose surjectivity in doing this. Then by linearity and the picture we have in our head of $[-\pi, \pi)$ mapping to a unit circle, we see right away that this map is injective.

2. Let us start by recalling that, given two *subsets* $X, Y \subset \mathbb{R}^n$ we say that they are diffeomorphic if there exist open neighbourhoods U, V (containing X, Y respectively) and a diffeomorphism $f : U \to V$ which is a bijection from X to Y.

In our case, the sets A, B, C, D are not smooth (sub)manifolds and it is implicit in the text of the problem that we should refer to the definition above. Let e_1, e_2 be the standard Euclidean basis of $\mathbb{R}^2$. For the first part, I claim we can actually find a *global* and *linear* diffeomorphism between A and B. Indeed we just need to pick a linear isomorphism $T : \mathbb{R}^2 \hookleftarrow$ such that the following three conditions are satisfied:

$$Te_1 = \lambda_1 e_1, \quad Te_2 = \lambda_2 e_2, \quad T(e_1 - e_2) = \lambda_3 (2e_1 - e_2) \text{ for } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\}.$$

Substituting the first two equations in the third one we only get the (linear) constraints

$$\lambda_1 = 2\lambda_3, \quad \lambda_2 = \lambda_3$$

and so we can pick, for instance, $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 1$. Therefore, in this basis we can identidify T with the 2×2 real diagonal matrix $diag\,(2, 1)$.

The second part is way more delicate and here is the structure of the argument. Let's argue by contradiction, so let's assume that a diffeomorphism $f : U \to V$ exists (for suitable U, V) so that $f(C) = D$ bijectively.

Step 1: $f(0) = 0$. This is shown by a topological argument, based on the notion of connected component. Indeed, f restricts to a *homeomorphism* between $C \setminus \{0\}$ and $D \setminus \{f(0)\}$ so that these two topological spaces must have, in particular, the same number of (path-)connected components. Now, $C \setminus \{0\}$ ha exactly 8 connected components, so $D \setminus \{f(0)\}$ must as well. The only admissible choice is then $f(0) = 0$ (since for any other assignment of $f(0)$ the set $D \setminus \{f(0)\}$ has only 2 (path-)connected components).

Step 2: reduction to the linear case. Let's take a 2nd order Taylor expansion of the map f centered at the origin: given Step 1, one gets that

$$f(0) = Df(0)x + O(|x|^2)$$

and so, as a result, one checks that $T := Df(0)$ must be a linear isomorphism $\mathbb{R}^2 \hookleftarrow$ sending C to D.

Step 3: direct treatment of the linear case. Lastly, we will find a contradiction by showing that there can't be any *linear* isomorphism from C to D. By possibly pre-composing with a rotation and an axial reflection, we can reduce to the case when T satisfies the four constraints:

$$Te_1 = \lambda_1 e_1, \quad Te_2 = \lambda_2 e_2, \quad T(e_1 - e_2) = \lambda_3 (2e_1 - e_2), \quad T(e_1 + e_2) = \lambda_4 (e_1 + e_2)$$

for $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \setminus \{0\}$.

This is only solved by $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ and this is a contradiction.

3. Change to polar coordinates. We have to chop up the domain into that part on which the argument of the absolute value is positive and that part on which it is negative.

$$\frac{x + y}{2} - x^2 - y^2 = r\left(\frac{\cos\theta + \sin\theta}{2} - r\right)$$

so the zero locus of the integrand is when $r = 0$ and when $r = (\cos\theta + \sin\theta)/2$. Thus, we split the integral J as follows:

$$J = \iint_D \left| \frac{x+y}{2} - x^2 - y^2 \right| dx, dy = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{(\sin\theta+\cos\theta)/2}^{1} r^2 \left(r - \frac{\cos\theta + \sin\theta}{2} \right) dr\, d\theta$$

$$+ \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{(\sin\theta+\cos\theta)/2} r^2 \left(\frac{\cos\theta + \sin\theta}{2} - r \right) dr\, d\theta + \int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}} \int_0^1 r^2 \left(r - \frac{\cos\theta + \sin\theta}{2} \right) dr\, d\theta$$

$$= (I) + (II) + (III).$$

Now by computing the integrals in dr and re-arranging one gets

$$J = \int_0^{2\pi} \left[\frac{1}{4} - \frac{\cos(\theta) + \sin(\theta)}{6} \right] d\theta + \frac{1}{96} \int_{-\pi/4}^{3\pi/4} (\cos(\theta) + \sin(\theta))^4 \, d\theta$$

so that

$$J = \frac{\pi}{2} + \frac{1}{96} \int_{-\pi/4}^{3\pi/4} (\cos(\theta) + \sin(\theta))^4 \, d\theta$$

but $\int_{-\pi/4}^{3\pi/4} (\cos(\theta) + \sin(\theta))^4 \, d\theta = 3\pi/2$ and hence

$$J = \frac{33\pi}{64}$$

4. Let us set, for clarity, $f(x, y) = x^{20}y^{13}$ and let us observe that $f(x, -y) = -f(x, y)$ for all $(x, y) \in D_a$. Since the domain D_a is itself symmetric with respect to the $x-$axis, I claim that $\int\int_{D_a} f(x, y)\, dxdy = 0$. This general statement has been proved in class, but let me repeat it here for completeness. Observe that

$$D_a = \{(x, y) \in \mathbb{R}^2 | -a \leq x \leq a, -h(x) \leq y \leq h(x)\}$$

where $h(x) = \sqrt{a^2 - x^2}$. Therefore, by Fubini's theorem:

$$\int\int_{D_a} f(x, y)\, dxdy = \int_{-a}^{a} \int_{-h(x)}^{h(x)} f(x, y) dydx = \int_{-a}^{a} \int_{-h(x)}^{0} f(x, y) dydx + \int_{-a}^{a} \int_{0}^{h(x)} f(x, y) dydx$$

and then, by changing variable in the first integral (setting $z = -y$) we get

$$\int_{-a}^{a} \int_{-h(x)}^{0} f(x, y) dydx = -\int_{-a}^{a} \int_{h(x)}^{0} f(x, -z) dzdx = \int_{-a}^{a} \int_{0}^{h(x)} f(x, -z) dzdx$$

hence

$$\int\int_{D_a} f(x,y)\,dxdy = \int_{-a}^{a}\int_{0}^{h(x)} [f(x,-y)+f(x,y)]\,dydx.$$

As a result, if $f(x,-y) = -f(x,y)$ at all points, we conclude that for all $-a \leq x \leq a$ indeed $\int_0^{h(x)} [f(x,-y)+f(x,y)]\,dy = 0$ and therefore $\int\int_{D_a} f(x,y)\,dxdy = 0$ as well.

5. Let R be the region in $\mathbb{R}^2$ described by

$$R = \{x \geq 0, y \geq 0, 1 \leq xy \leq 2, x \leq y \leq 2x\}.$$

Then

$$\mathrm{Vol}(U) = \iint_R (x+y)\,dx\,dy.$$

It is easier to imagine R and make up a nice change of variables than it is to do so with U. The defining equations suggest

$$\begin{aligned} r &= xy \\ \theta &= y/x, \end{aligned}$$

but actually to apply the change of variables theorem, we need the map going in the other direction, so invert the map:

$$\begin{aligned} x &= \sqrt{\frac{r}{\theta}} \\ y &= \sqrt{r\theta} \end{aligned}$$

Then R pulls back to the region $[1,2]\times[1,2]$ in the (r,θ)-plane. The Jacobian is

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{r\theta}} & \frac{-\sqrt{r}}{2\theta^{3/2}} \\ \frac{\sqrt{\theta}}{2\sqrt{r}} & \frac{\sqrt{r}}{2\sqrt{\theta}} \end{pmatrix}.$$

So, the absolute value of the determinant of the Jacobian comes out to be $\frac{1}{2\theta}$. Finally, we compute

$$\begin{aligned} \mathrm{Vol}(U) &= \iint_R (x+y)\,dx\,dy \\ &= \int_1^2\int_1^2 \left(\frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}\right)\frac{\sqrt{r}}{2}\,dr\,d\theta \\ &= \frac{2\sqrt{2}-1}{3}\int_1^2 \frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}\,d\theta \\ &= \frac{4-\sqrt{2}}{3}. \end{aligned}$$