

Math 52H Homework 5 Solutions

February 12, 2013

1. a. The integral is

$$\begin{aligned}\int_0^1 \int_{2y}^{y+1} x - y dx dy &= \int_0^1 \frac{(y+1)^2}{2} - (y+1)y - \frac{(2y)^2}{2} + 2y^2 dy \\ &= \int_0^1 \frac{-y^2 + 1}{2} dy \\ &= 1/2 - 1/6 \\ &= 1/3.\end{aligned}$$

For a, b, c either 0 or 1, the rest of the questions all integrate

$$\begin{aligned}\int_B x^a y^b z^c dV &= \int_0^1 \int_0^1 \int_0^{xy} x^a y^b z^c dz dx dy \\ &= \int_0^1 \int_0^1 x^a y^b (xy)^{c+1} / (c+1) dx dy = \frac{1}{(c+1)(a+c+2)(b+c+2)}.\end{aligned}$$

The numerical answers follow from the above formula, and are (in order) $1/6, 1/6, 1/18,$ and $1/9$.

2. In these sorts of problems it is often convenient to pick a set A which is in fact a product $\Gamma \times \Lambda$ for $\Gamma, \Lambda \subset \mathbb{R}$. Let us then follow this *Ansatz* and pick $\Lambda \subset [0, 1]$ to be *any* non-Riemann measurable subset of the unit interval. For instance, one might take $\Lambda = \mathbb{Q} \cap [0, 1]$. The fact that such Λ is not measurable (in the sense of Riemann) follows at one from Corollary 9.14 of your notes: indeed if it were, then $\text{Int}(\Lambda) = \emptyset$ and $\overline{\Lambda} = [0, 1]$ would be as well and would have the same measure, which gives a contradiction. As a second step, let us pick $\Gamma = \overline{\{\frac{1}{n} | n \in \mathbb{N}\}}$ and let us define $A = \Gamma \times \Lambda$. Clearly, for $x = 1/n$ the section $A_{1/n}$ is not Riemann measurable (in the line), while I claim that the set A is measurable as a subset of

\mathbb{R}^2 . In the Riemann theory, we can argue as follows: we need to show that the function χ_A is integrable by using lower and upper approximations. To get a lower approximation, let us pick the *trivial partition* \mathcal{P}_l simply given by $[0, 1] \times [0, 1]$ so that $L(\chi_A; \mathcal{P}_l) = 0$ and hence, as a trivial consequence $\sup_{\mathcal{P}} L(\chi_A; \mathcal{P}) \geq 0$. Therefore, we only need to show that $\inf_{\mathcal{P}} U(\chi_A; \mathcal{P}) \leq 0$. To that aim, for $m \in \mathbb{N}$ pick a partition \mathcal{P}_m determined by the vertices $\{t_j\} \times \{0\}$ and $\{t_j\} \times \{1\}$ where

$$t_0 = 0, t_1 = \frac{1}{m}, t_{2m} = 1, t_{2j} = \frac{1}{m-j} - 2^{-(m+1)}, t_{2j+1} = \frac{1}{m-j} + 2^{-(m+1)} \quad \text{for } 1 \leq j < m.$$

In other terms, this partition is trivial on the second factor and is made of finitely many rectangles of the form $[t_j, t_{j+1}] \times [0, 1]$. It is readily checked that $U(\chi_A; \mathcal{P}_m) = \frac{1}{m} + \frac{m}{2^m}$ so that

$$\inf_{\mathcal{P}} U(\chi_A; \mathcal{P}) \leq \lim_{m \rightarrow \infty} U(\chi_A; \mathcal{P}_m) = 0$$

and this completes the proof.

3. a. Using polar coordinates, the given bound translates to $r^2 \leq 2r \cos \theta$, whence $0 \leq r \leq 2 \cos \theta$. Note that this implies that $\cos \theta \geq 0$, so that $-\pi/2 \leq \theta \leq \pi/2$. (Alternatively, you can complete the square to see that the equation of the circle is $(x-1)^2 + y^2 = 1$, which is on the right side of the y-axis.) We get that the integral is

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^3 dr d\theta &= 4 \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta \\ &= 4 \int_{-\pi/2}^{\pi/2} \cos^2 \theta (1 - \sin^2 \theta) d\theta \\ &= 4 \int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2} d\theta - \int_{-\pi/2}^{\pi/2} \sin^2 2\theta d\theta \\ &= 2\pi - \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 4\theta}{2} d\theta \\ &= \frac{3\pi}{2}. \end{aligned}$$

b. Change coordinates to $u = x/a$ and $v = y/b$ which has Jacobian ab . The integral then becomes $ab \int_D \sqrt{1-u^2-v^2} dV$ where $D = \{u^2 + v^2 \leq 1\}$. Now change to polar coordinates to get

$$\begin{aligned} ab \int_0^{2\pi} \int_0^1 \sqrt{1-r^2} r dr d\theta &= ab \int_0^{2\pi} 1 d\theta [-1/3(1-r^2)^{3/2}]_0^1 \\ &= 2ab\pi/3. \end{aligned}$$

4. Let the region in question be D . We calculate $\frac{\partial u}{\partial x} = \frac{2x}{y}$, $\frac{\partial u}{\partial y} = -\frac{x^2}{y^2}$, $\frac{\partial v}{\partial x} = -\frac{y^2}{x^2}$, and $\frac{\partial v}{\partial y} = \frac{2y}{x}$. The Jacobian determinant is $\frac{\partial(u,v)}{\partial(x,y)} = 3$. Thus we have $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}$. The bounds given in the question translate to $a \leq u \leq b$, and $\alpha \leq v \leq \beta$.

Thus, the area of D is

$$\int_D 1dV = \frac{1}{3} \int_a^b \int_\alpha^\beta 1dvdu = \frac{(b-a)(\beta-\alpha)}{3}.$$

5. Use cylindrical coordinates, so that the hyperboloid becomes $r^2 - z^2 = a^2$, and the region bounded by the sphere is described by $r^2 + z^2 \leq 3a^2$. We calculate the volume of the region given by $r^2 - z^2 \geq a^2$. These bounds imply that $a \leq r \leq \sqrt{3}a$. We divide this into two regions. Let A_1 denote the area of region given by $a \leq r \leq \sqrt{2}a$, and A_2 denote the area of the region given by $\sqrt{2}a \leq r \leq \sqrt{3}a$. Then, by symmetry

$$\begin{aligned} \frac{A_1}{2} &= \int_0^{2\pi} \int_a^{\sqrt{2}a} \int_0^{\sqrt{r^2-a^2}} r dz dr d\theta \\ &= 2\pi \int_a^{\sqrt{2}a} \sqrt{r^2-a^2} r dr \\ &= 2\pi [(r^2-a^2)^{3/2}/3]_a^{\sqrt{2}a} \\ &= \frac{2\pi a^3}{3}. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{A_2}{2} &= \int_0^{2\pi} \int_{\sqrt{2}a}^{\sqrt{3}a} \int_0^{\sqrt{3a^2-r^2}} r dz dr d\theta \\ &= 2\pi \int_{\sqrt{2}a}^{\sqrt{3}a} \sqrt{3a^2-r^2} r dr \\ &= 2\pi [-(3a^2-r^2)^{3/2}/3]_{\sqrt{2}a}^{\sqrt{3}a} \\ &= \frac{2\pi a^3}{3}. \end{aligned}$$

As a fraction of the total volume of the ball, this is

$$\frac{A_1 + A_2}{4/3\pi(\sqrt{3}a)^3} = \frac{2}{3\sqrt{3}}.$$

The ratio is therefore $2 : (3\sqrt{3} - 2)$.

6. We prove the contrapositive: Assume that $\text{Vol } A > 0$ and that A is Riemann measurable. We show that $\text{Int } A$ is nonempty. From our definition of interior, it suffices to show that there is some open set contained in A . Because the appropriate context for Riemann integration is the real line, we assume for this problem that $A \subset \mathbb{R}$. A set is Riemann measurable if the indicator function $\chi_A(x)$ is Riemann integrable. Let \mathcal{P} be any partition of A . Then we have that $\sup_{\mathcal{P}} \mathcal{L}(\chi_A, \mathcal{P}) = \inf_{\mathcal{P}} \mathcal{U}(\chi_A, \mathcal{P}) > 0$. So, this says that there exists a partition \mathcal{P}_0 so that

$$\sum_{\mathcal{P}} \inf_{x_i \in [t_i, t_{i+1}]} \chi_A(x_i)(t_{i+1} - t_i) > 0.$$

Thus, by positivity, there must be some closed interval $[t_i, t_{i+1}]$ for which $\inf_{x_i \in [t_i, t_{i+1}]} \chi_A(x_i) > 0$, i.e. $[t_i, t_{i+1}] \subset A$. So A contains the open set (t_i, t_{i+1}) . So it has nonempty interior, and is not nowhere dense. Thus we have shown the first part.

Secondly, take a “positive measure cantor set”, say \mathcal{K} as an example. To construct \mathcal{K} , take \mathcal{K}_1 to be $[0, 1/3] \cup [2/3, 1]$, i.e. remove the middle $1/3$. At the second stage, we remove the middle $1/9$:

$$\mathcal{K}_2 = [0, 4/27] \cup [5/27, 9/27] \cup [18/27, 22/27] \cup [23/27, 1].$$

At the \mathcal{K}_3 stage, remove the middle $1/27$ of every remaining interval.

Similarly to the usual Cantor set, \mathcal{K} is an intersection of closed sets, and hence is closed. Thus \mathcal{K} is its own closure. Suppose \mathcal{K} contained some open interval U . Then U must be contained in every \mathcal{K}_n . But U is connected, so must lie entirely in only one of the connected components of \mathcal{K}_n for any n . But any open interval has some length ℓ . Take n so large so that

$$\prod_{i=1}^n \left(1 - \frac{1}{3^i}\right) / 2^n < \ell,$$

i.e. each of the subintervals of \mathcal{K}_n are shorter than U . Contradiction. Thus \mathcal{K} contains no open sets, and hence is nowhere dense.

Now we show that \mathcal{K} is not Riemann measurable. On the one hand, \mathcal{K} contains no open intervals, hence contains no closed intervals, hence

$$\mathcal{L}(\chi_{\mathcal{K}}, \mathcal{P}) := \sum_{\mathcal{P}} \inf_{x_i \in [t_i, t_{i+1}]} \chi_{\mathcal{K}}(x_i)(t_{i+1} - t_i) = 0$$

for all partitions! So we have that $\sup_{\mathcal{P}} \mathcal{L}(\chi_{\mathcal{K}}, \mathcal{P}) \leq 0$. On the other hand, we see that the total length of \mathcal{K} is given by

$$\prod_{i=1}^{\infty} \left(1 - \frac{1}{3^i}\right) > 0,$$

so that

$$\mathcal{U}(\chi_{\mathcal{K}}, \mathcal{P}) := \sum_{\mathcal{P}} \sup_{x_i \in [t_i, t_{i+1}]} \chi_{\mathcal{K}}(x_i)(t_{i+1} - t_i) \geq \prod_{i=1}^{\infty} \left(1 - \frac{1}{3^i}\right)$$

for all partitions. Hence, we have that $\inf_{\mathcal{P}} \mathcal{U}(\chi_{\mathcal{K}}, \mathcal{P}) \geq \prod_{i=1}^{\infty} \left(1 - \frac{1}{3^i}\right)$. Thus, $\chi_{\mathcal{K}}$ is not Riemann integrable, i.e. \mathcal{K} is not Riemann measurable.