Math 52H Homework 4 Solutions

February 11, 2013

1. We start recalling the definition of the divergence operator given in the Lecture Notes in Section 7.6:

$$\operatorname{div}(v) = \Lambda^{-1}\left(d\left(\lrcorner v\right)\right).$$

The first step is to express the standard volume form $\Omega = dx \wedge dy$ in polar coordinates: since by definition $x = r\cos(\phi)$ and $y = r\sin(\phi)$ by differentiating and taking the wedge product we get at once $\Omega = rdr \wedge d\phi$.
As a result, we can compute $\Omega \lrcorner v = \Omega\left(v, \cdot\right)$ namely

$$\Omega \lrcorner v = \left(dr \otimes d\phi - d\phi \otimes dr\right)\left(a\mathbf{e}_r + b\mathbf{e}_\phi, \cdot\right) = ard\phi - bdr.$$

Therefore, by taking the differential

$$d\left(\lrcorner v\right) = \left[\frac{\partial\left(ar\right)}{\partial r} + \frac{\partial b}{\partial \theta}\right] dr \wedge d\theta$$

and so, applying the isomorphism Λ^{-1} (which basically amounts to diving by r since $\Omega = rdr \wedge d\theta$ as we saw before) we end up getting

$$\operatorname{div}(v) = \frac{1}{r}\left[\frac{\partial\left(ar\right)}{\partial r} + \frac{\partial b}{\partial \theta}\right].$$

2. Let $f = \sum_{i=1}^{n} x_i x_{i+n}$ and let $\mathcal{V} = dx_1 \wedge ... \wedge dx_{2n}$ denote the volume form. Then

$$\begin{aligned}
d\Omega &= \frac{d(\omega \wedge \theta)}{f^n} - n\sum_{1}^{n}(x_{i+n}dx_i + x_i dx_{i+n}) \wedge \frac{\omega \wedge \theta}{f^{n+1}} \\
&= \frac{n}{f^n}\left(\mathcal{V} - \sum_{1}^{n}(x_{i+n}dx_i + x_i dx_{i+n}) \wedge \frac{\omega \wedge \theta}{f}\right) \\
&= \frac{n}{f^n}\left(\mathcal{V} - \sum_{1}^{n} x_i x_{i+n}\frac{\mathcal{V}}{f}\right) \\
&= 0,
\end{aligned}$$

as desired.

3. We have $\Delta = \partial^2 + \partial d + d\partial + d^2$. Now

$$\partial^2 = \star^{-1} d \star \star^{-1} d\star = \star^{-1} dd\star = 0$$

so $\Delta = \partial d + d\partial$ and this is a map $\Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$.

We compute $\Delta(\alpha)$ for an arbitrary 1-form $\alpha(x) = \alpha_1(x)dx_1 + ... + \alpha_n(x)dx_n$. So that we don't confuse the ∂ operator with plain-old partial derivatives $\frac{\partial}{\partial x_i}$ we'll use the alternative notation $D_i = \frac{\partial}{\partial x_i}$ in this problem. We have

$$\begin{aligned}
\partial d\alpha &= \partial \left(\sum_{i \neq j} D_j \alpha_i dx_j \wedge dx_i \right) \\
&= \partial \left[\sum_{i<j} (D_i \alpha_j - D_j \alpha_i)\, dx_i \wedge dx_j \right] \\
&= (-1)^{3n+1} \star d \left[\sum_{i<j} (D_i \alpha_j - D_j \alpha_i) \star (dx_i \wedge dx_j) \right] \\
&= (-1)^{n+1} \star \left[\sum_{i<j} \left\{ \left(D_i^2 \alpha_j - D_i D_j \alpha_i\right) dx_i \wedge \star(dx_i \wedge dx_j) + \left(D_i D_j \alpha_j - D_j^2 \alpha_i\right) dx_j \wedge \star(dx_i \wedge dx_j) \right\} \right]
\end{aligned}$$

Now $dx_i \wedge \star(dx_i \wedge dx_j) = \pm \star dx_j$. We can determine the sign by noting (ω is the volume form)

$$dx_i \wedge dx_j \wedge \star(dx_i \wedge dx_j) = \omega = dx_j \wedge \star dx_j.$$

Hence $dx_j \wedge dx_i \wedge \star(dx_i \wedge dx_j) = -\omega$ so $dx_i \wedge \star(dx_i \wedge dx_j) = - \star dx_j$. Similarly we find $dx_j \wedge \star(dx_i \wedge dx_j) = \star dx_i$. Plugging this in,

$$\begin{aligned}
\partial d\alpha &= (-1)^{n+1} \star \left[\sum_{i<j} \left\{ \left(D_i D_j \alpha_i - D_i^2 \alpha_j\right) \star dx_j + \left(D_i D_j \alpha_j - D_j^2 \alpha_i\right) \star dx_i \right\} \right] \\
&= (-1)^{n+1+n-1} \sum_{i<j} \left\{ \left(D_i D_j \alpha_i - D_i^2 \alpha_j\right) dx_j + \left(D_i D_j \alpha_j - D_j^2 \alpha_i\right) dx_i \right\} \\
&= \sum_{i \neq j} (D_i D_j \alpha_j - D_j^2 \alpha_i) dx_i.
\end{aligned}$$

We also have

$$\begin{aligned} d\partial\alpha &= (-1)^{2n+1} d \star d\left(\sum_{i=1}^{n} \alpha_i \star dx_i\right) = -d\star\left(\sum_{i=1}^{n} D_i\alpha_i dx_i \wedge \star dx_i\right) \\ &= -d\star\left(\sum_{i=1}^{n} D_i\alpha_i\omega\right) = -d\left(\sum_{i=1}^{n} D_i\alpha_i\right) \\ &= -\sum_{i,j=1}^{n} D_iD_j\alpha_i dx_j \end{aligned}$$

Hence

$$\Delta\alpha = (d\partial + \partial d)\alpha = \sum_{i\neq j}(D_iD_j\alpha_j - D_j^2\alpha_i)dx_i - \sum_{i,j} D_iD_j\alpha_j dx_i = -\sum_{i,j} D_j^2\alpha_i dx_i.$$

It follows that

$$\Delta(f\alpha) = -\sum_{i,j} D_j^2(f\alpha_i)dx_i = -\sum_{i,j}(\alpha_i D_j^2 f + 2D_jfD_j\alpha_i + fD_j^2\alpha_i)dx_i.$$

Let's agree to write ∇^2 for the second order differential operator $\nabla^2 = \sum_{j=1}^n D_j^2$. Thus our calculation from part b shows that for 1-form $\alpha = \alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dx_3$ we have

$$\Delta\alpha = -(\nabla^2\alpha_1 dx_1 + \nabla^2\alpha_2 dx_2 + \nabla^2\alpha_3 dx_3).$$

For α a zero form, $\partial\alpha = \pm \star d \star \alpha = 0$ since $\star\alpha$ is an n-form. Hence

$$\begin{aligned} \Delta\alpha = (d\partial + \partial d)\alpha = \partial d\alpha &= (-1)^{2n+1} \star d(D_1\alpha dx_2 \wedge dx_3 + D_2\alpha dx_3 \wedge dx_1 + D_3\alpha dx_1 \wedge dx_2) \\ &= -(D_1^2\alpha + D_2^2\alpha + D_3^2\alpha) = -\nabla^2\alpha. \end{aligned}$$

Now the trick to doing the $k = 2, 3$ cases without more horrible computation is the following

Lemma 1. *Δ commutes with the Hodge star operator operator $\star$, i.e. $\star\Delta\alpha = \Delta\star\alpha$.*

Proof. Indeed, $\star\Delta\alpha = \star(d\partial + \partial d) = -(\star d \star^{-1} d \star + d \star d)$ while $\Delta\star\alpha = (d\partial + \partial d)\star = -(d \star^{-1} d \star\star + \star^{-1} d \star d\star)$. The two are equal since $\star^{-1} = (-1)^{k(n-k)}\star$, and $\star\star = (-1)^{k(n-k)}\mathrm{Id}$. □

This lemma allows us to reduce the cases of $k = 2$ and $k = 3$ to 1-forms and 0-forms, respectively. Essentially, we can use the $\star$ operator to flip a 2-form into a 1-form, apply the Laplace-de Rham operator there, and then use $\star^{-1}$ to flip back to a 2-form. We get that for α the 2-form

$$\alpha = \alpha_1 dx_2 \wedge dx_3 + \alpha_2 dx_3 \wedge dx_1 + \alpha_3 dx_1 \wedge dx_2,$$

$$\Delta\alpha = -(\nabla^2\alpha_1 dx_2 \wedge dx_3 + \nabla^2\alpha_2 dx_3 \wedge dx_1 + \nabla^2\alpha_3 dx_1 \wedge dx_2).$$

Lastly, for 3-forms we have that

$$\Delta\alpha(x) dx_1 \wedge dx_2 \wedge dx_3 = -\nabla^2\alpha dx_1 \wedge dx_2 \wedge dx_3.$$