Math 52H Homework 3 Solutions

February 1, 2013

1. There are $\binom{4}{3} = 4$ distinct 3-dimensional faces, one for each choice of 3 vectors which span the face. To compute the volumes of the faces, we apply Proposition 3.3 of §3.4 of the notes. Let P_{v_1,v_2,v_3} be the face spanned by the vectors v_1, v_2, v_3. We have that $(Vol_k P_{v_1,v_2,v_3})^2 = \det G(v_1, v_2, v_3)$, where G is the Gram matrix

$$G(v_1, v_2, v_3) = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \langle v_1, v_3\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \langle v_2, v_3\rangle \\ \langle v_3, v_1\rangle & \langle v_3, v_2\rangle & \langle v_3, v_3\rangle \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

So a little computation yields

$$Vol_k P_{v_1,v_2,v_3} = 4\sqrt{2}.$$

By symmetry, we have that

$$Vol_k P_{v_1,v_2,v_3} = Vol_k P_{v_1,v_2,v_4} = Vol_k P_{v_1,v_3,v_4},$$

and lastly,

$$G(v_2, v_3, v_4) = \begin{pmatrix} \langle v_2, v_2\rangle & \langle v_2, v_3\rangle & \langle v_2, v_4\rangle \\ \langle v_3, v_2\rangle & \langle v_3, v_3\rangle & \langle v_3, v_4\rangle \\ \langle v_4, v_2\rangle & \langle v_4, v_3\rangle & \langle v_4, v_4\rangle \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

and so computing the determinant,

$$Vol_k P_{v_2,v_3,v_4} = 8.$$

Lastly, the change of basis matrix from e_1, e_2, e_3, e_4 to v_1, v_2, v_3, v_4 is given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix},$$

which has determinant $= -8$, so the basis defined by v_1, v_2, v_3, v_4 is oppositely oriented to the standard basis.

2. First of all, we assume that $c \neq 0$. I claim that

Lemma 1. *Let $\omega \neq 0$ be a k-form on V. Let*

$$L_\omega := \{v \in V | \omega(v, x_2, \dots, x_k) = 0 \ \forall x_2, \dots, x_k \in V\}.$$

If $\ell_1, \dots, \ell_k$ are linearly independent 1-forms, and we set $\omega = \ell_1 \wedge \cdots \wedge \ell_k$, then $\dim L_\omega = n - k$, and moreover,

$$L_\omega = \{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\}.$$

Proof. We show that for this choice of ω we have $\dim L_\omega \leq n - k$. Because $\omega \neq 0$, we have that for fixed $x_2, \dots, x_k$, $\omega(v, x_2, \dots, x_k)$ is a nonzero linear function of v, hence, has kernel dimension $n-1$. I claim that if x_2' is linearly independent from then $x_2, \dots, x_k$, then $\ker \omega(-, x_2', \dots, x_k)$ is independent from $\ker \omega(-, x_2, \dots, x_k)$. Indeed this is true because $\omega(v, x_2, \dots, x_k)$ and $\omega(v, x_2', \dots, x_k)$ are linearly independent as linear functions of v. Then we have that

$$\dim\{v \in V | \omega(v, x_2, \dots, x_k) = 0 \ \forall x_2, \dots, x_k \in V\} \leq n - 2.$$

Repeat this process for each entry to find that in fact the dimension is $\leq n - k$.

On the other hand, $\{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\}$ is clearly $n - k$ dimensional because the $\ell_1, \dots, \ell_k$ are linearly independent, and

$$\{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\} \subset L_\omega,$$

hence $\dim L_\omega \geq n - k$ and the two sets are in fact equal. $\square$

We have that by assumption

$$\mathcal{A}^*\ell_1 \wedge \cdots \wedge \mathcal{A}^*\ell_k = \mathcal{A}^*(\ell_1 \wedge \cdots \wedge \ell_k) = c\ell_1 \wedge \cdots \wedge \ell_k,$$

so that by the lemma we have

$$\{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\} = L_{\ell_1 \wedge \cdots \wedge \ell_k} = L_{\mathcal{A}^*\ell_1 \wedge \cdots \wedge \mathcal{A}^*\ell_k} = \{v \in V | \mathcal{A}^*\ell_1(v) = \cdots = \mathcal{A}^*\ell_k(v) = 0\}.$$

Suppose now that for some i, $\mathcal{A}^*\ell_i \notin \operatorname{span}(\ell_1, \dots, \ell_k)$. For any nonzero $\lambda \notin \operatorname{span}(\ell_1, \dots, \ell_k)$, we have that $\{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\} \not\subset \{v \in V | \lambda(v) = 0\}$. On the other hand, we also have that $\{v \in V | \mathcal{A}^*\ell_1(v) = \cdots = \mathcal{A}^*\ell_k(v) = 0\} \subset \{v \in V | \mathcal{A}^*\ell_i(v) = 0\}$. Thus we have reached a contradiction, and we must have $\mathcal{A}^*\ell_i \in \operatorname{span}(\ell_1, \dots, \ell_k)$ for all $i = 1, \dots, k$.

3. For $i = 1, 2, \ldots, n$ let us set $Q_i(x_1, \ldots, x_n) = x_i$ and hence let us define $P_i(x_1, \ldots, x_n) = D(Q_i)$. Since by assumption $D : \mathcal{P} \to \mathcal{P}$ (namely the operator D maps polynomials into polynomials) we have that for each choice of the index $i = 1, 2, \ldots, n$ the function P_i is in fact a polynomial. Then, let us define the operator $T : \mathcal{P} \to \mathcal{P}$ by the formula $TP = \sum_{i=1}^{n} P_i \frac{\partial P}{\partial x_i}$ and all the problem is reduced to showing that in fact $D = T$ on $\mathcal{P}$. To that aim, let us start by observing that such T is linear and satisfies the Leibniz rule (as it is for D, by assumption). Let us also observe that for any $c \in \mathbb{R}$ (viewed as a degree 0 polynomial) we have $Tc = Dc = 0$ (since $D(c) = D(c \cdot 0) = D(c) \cdot 0 + c \cdot D(0) = 0$ where we have used both the Leibniz property and the linearity of D).
Now, we check that $T = D$ on $\mathcal{P}$ proceeding by induction on the degree. By the previous remark and by construction the claim is true on $\mathcal{P}_{\deg \leq 1}$ and then given $D = T$ on $\mathcal{P}_{\deg \leq n}$ we get at once that $D = T$ on $\mathcal{P}_{\deg \leq (n+1)}$ using the Leibniz rule. Indeed: given a monomial p of degree $n + 1$ let us split it as $p = qr$ with $\deg(q) \leq n$ (and so $\deg(r) \leq n$ as well) and so

$$D(p) = D(qr) = rD(q) + qD(r) = rT(q) + qT(r) = T(qr) = T(p).$$

After having proved the claim for monomials of degree $n + 1$, the case of an arbitrary polynomial of degree $n + 1$ follows by linearity of both T and D.

4. First, we check that

$$df = \frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2.$$

Indeed, $d_y f \in V_y^*$ is a linear function on the tangent space, so for any $h = h_1 v_1 + h_2 v_2 \in V_y$, we have

$$d_y f(h) = h_1 d_y f(v_1) + h_2 d_y f(v_2) = \frac{\partial f}{\partial y_1}(y) h_1 + \frac{\partial f}{\partial y_2}(y) h_2$$

by the second displayed equation of page 46 of the notes, §5.1. Also, if y_1, y_2 are the standard coordinate functions on V, which we now think of as smooth functions on V, then we have by linearity

$$dy_i = \lim_{t \to 0} \frac{y_i(y + th) - y_i(y)}{t} = y_i(h) = h_i,$$

so we get the formula for the differential we wanted.

Now we find the formula for the gradient. We have $\nabla f(y) = \mathcal{D}^{-1}(d_y f)$. Hence for any $h = h_1 v_1 + h_2 v_2 \in V_y$

$$h_1 \frac{\partial f}{\partial y_1}(y) + h_2 \frac{\partial f}{\partial y_2}(y) = \langle \nabla f(y), h \rangle = \langle g_1(y) v_1 + g_2(y) v_2, h_1 v_1 + h_2 v_2 \rangle,$$

or using symmetry and linearity of inner products,

$$\left[\frac{\partial f}{\partial y_1}(y), \frac{\partial f}{\partial y_2}(y) \right] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = [g_1(y), g_2(y)] \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix},$$

so, since h was arbitrary,

$$[g_1(y), g_2(y)] = \left[\frac{\partial f}{\partial y_1}(y), \frac{\partial f}{\partial y_2}(y)\right] \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \left[2\frac{\partial f}{\partial y_1}(y) - \frac{\partial f}{\partial y_2}(y), -\frac{\partial f}{\partial y_1}(y) + \frac{\partial f}{\partial y_2}(y)\right],$$

so we have

$$\nabla f(y) = \left(2\frac{\partial f}{\partial y_1}(y) - \frac{\partial f}{\partial y_2}(y)\right)\frac{\partial}{\partial y_1} + \left(-\frac{\partial f}{\partial y_1}(y) + \frac{\partial f}{\partial y_2}(y)\right)\frac{\partial}{\partial y_2}.$$

5. We have

$$\begin{aligned} dx &= \sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta \\ dy &= \sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta \\ dz &= \cos\phi d\rho - \rho\sin\phi d\phi. \end{aligned}$$

Hence the 1-form is equal to

$$\begin{aligned} &\cos\phi d\rho - \rho\sin\phi d\phi + \frac{1}{2}\Big(\rho\sin\phi\cos\theta\left[\sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta\right] \\ &\qquad\qquad - \rho\sin\phi\sin\theta\left[\sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta\right]\Big) \\ &= \cos\phi d\rho - \rho\sin\phi d\phi + \frac{1}{2}\rho^2\sin^2\phi d\theta. \end{aligned}$$

6. Since ω is closed,

$$d\omega = (\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1)dx\wedge dy + (\frac{\partial}{\partial z}F_1 - \frac{\partial}{\partial x}F_3)dz\wedge dx + (\frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2)dy\wedge dz = 0$$

so we have the three equalities

$$\frac{\partial}{\partial x}F_2 = \frac{\partial}{\partial y}F_1, \quad \frac{\partial}{\partial z}F_1 = \frac{\partial}{\partial x}F_3, \quad \frac{\partial}{\partial y}F_3 = \frac{\partial}{\partial z}F_2. \tag{1}$$

Differentiating the homogeneity equation with respect to t, and then setting $t = 1$ we obtain

$$F_k(x,y,z) = x\frac{\partial}{\partial x}F_k(x,y,z) + y\frac{\partial}{\partial y}F_k(x,y,z) + z\frac{\partial}{\partial z}F_k(x,y,z). \tag{2}$$

Hence

$$\begin{aligned} df &= \frac{1}{2}\Big((F_1 + x\frac{\partial}{\partial x}F_1 + y\frac{\partial}{\partial x}F_2 + z\frac{\partial}{\partial x}F_3)dx + (x\frac{\partial}{\partial y}F_1 + F_2 + y\frac{\partial}{\partial y}F_2 + z\frac{\partial}{\partial y}F_3)dy \\ &\qquad + (x\frac{\partial}{\partial z}F_1 + y\frac{\partial}{\partial z}F_2 + F_3 + z\frac{\partial}{\partial z}F_3)dz\Big) \\ &= \frac{1}{2}\omega + \frac{1}{2}\Big((x\frac{\partial}{\partial x}F_1 + y\frac{\partial}{\partial y}F_1 + z\frac{\partial}{\partial z}F_1)dx + (x\frac{\partial}{\partial x}F_2 + y\frac{\partial}{\partial y}F_2 + z\frac{\partial}{\partial z}F_2)dy \\ &\qquad + (x\frac{\partial}{\partial x}F_3 + y\frac{\partial}{\partial y}F_3 + z\frac{\partial}{\partial z}F_3)dz\Big) \\ &= \omega. \end{aligned}$$