Math 52H Homework 2 Solutions

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1. (a). We can check this straight from the definitions. Let $\alpha \in \Lambda^k(V^*)$ be a k-form on V. Let $U_1, \dots, U_{n-k}$ be any vectors. First, assume that they are linearly dependent. Then clearly $(\mathcal{A}^* \circ \star)\alpha(U_1, \dots, U_{n-k}) = 0$ because $\star\alpha(U_1, \dots, U_{n-k}) = 0$. But if we call $\omega = \mathcal{A}^*\alpha$, then $\star \circ \mathcal{A}^*\alpha(U_1, \dots, U_{n-k}) = \star\omega(U_1, \dots, U_{n-k}) = 0$ by definition of the Hodge-$\star$ operator. So we have shown the identity in the case that the vectors are linearly dependent.

Now assume $U_1, \dots, U_{n-k}$ are linearly independent. Let $Z_1, \dots, Z_k$ be the complementary vectors in the sense of Definition 4.1 in the notes. Then $\mathcal{A}^* \circ \star\alpha(U_1, \dots, U_{n-k}) = \star\alpha(\mathcal{A}U_1, \dots, \mathcal{A}U_{n-k})$. Now, because $\mathcal{A}$ is special orthogonal, we have that $\mathcal{A}Z_1, \dots, \mathcal{A}Z_k$ is a basis of $\operatorname{span}(\mathcal{A}U_1, \dots, \mathcal{A}U_{n-k})^\perp$ with the same volume and the same orientation. Thus

$$\mathcal{A}^* \circ \star\alpha(U_1, \dots, U_{n-k}) = \star\alpha(\mathcal{A}U_1, \dots, \mathcal{A}U_{n-k}) = \alpha(\mathcal{A}Z_1, \dots, \mathcal{A}Z_k).$$

On the other hand, if we call $\omega := \mathcal{A}^*\alpha$, then

$$\star \circ \mathcal{A}^*\alpha(U_1, \dots, U_{n-k}) = \star\omega(U_1, \dots, U_{n-k}) = \omega(Z_1, \dots, Z_k) = \alpha(\mathcal{A}Z_1, \dots, \mathcal{A}Z_k).$$

(b). Choose an orthonormal basis $e_1, \dots, e_n$ and apply the previous to the k-form $x_{i_1} \wedge \cdots \wedge x_{i_k}$. Let $j_1, \dots j_{n-k}$ be some choice of $n-k$ indices, and let $\widehat{j}_1, \dots, \widehat{j}_k$ be the complementary k indices. Likewise, let $\widehat{i}_1, \dots, \widehat{i}_{n-k}$ be the complementary indices to the is. Then by the previous part,

$$(\mathcal{A}^* \circ \star)x_{i_1} \wedge \cdots \wedge x_{i_k}(e_{j_1}, \dots e_{j_{n-k}}) = x_{i_1} \wedge \cdots \wedge x_{i_k}(Ae_{\widehat{j}_1}, \dots, Ae_{\widehat{j}_k}),$$

which by, say, Prop 2.17 from the course notes is the minor of A given by the $i_1, \dots, i_k$ and the $\widehat{j}_1, \dots, \widehat{j}_k$. On the other hand, this is equal to

$$\begin{aligned}(\star \circ \mathcal{A}^*)x_{i_1} \wedge \cdots \wedge x_{i_k}(e_{j_1}, \dots, e_{j_{n-k}}) &= \star x_{i_1} \wedge \cdots \wedge x_{i_k}(Ae_{j_1}, \dots, Ae_{j_{n-k}}) \\ &= (-1)^{\operatorname{inv}(i_1, \dots, i_k, \widehat{i}_1, \dots, \widehat{i}_{n-k})} x_{\widehat{i}_1} \wedge \cdots \wedge x_{\widehat{i}_{n-k}}(Ae_{j_1}, \dots, Ae_{j_{n-k}}),\end{aligned}$$

which is up to a ± 1 the minor of A given by the $\widehat{i}_1, \dots, \widehat{i}_k$, and the $j_1, \dots, j_{n-k}$. So the two minors are the same in absolute value.

2.

b. Let $e_1, ..., e_{2n}$ be basis vectors for $\mathbb{R}^{2n}$, where e_{2k-1} corresponds to x_k and e_{2k} corresponds to y_k for $1 \le k \le n$. We first find the matrix corresponding to the skew-symmetric bilinear form ω. Say that $i \le j$. Then, we have that $\omega(e_i, e_j) = 0$ unless $i = 2k - 1$ and $j = 2k$ for some k in which case $\omega(e_i, e_j) = 1$. By skew-symmetry of ω, the matrix corresponding to ω is $-J$. Now $\mathcal{A}^*\omega = \omega$ implies that for $u, v \in \mathbb{R}^{2n}$ that

$$u^t(-J)v = u^t A^t(-J)Av,$$

which immediately gives $J = A^t JA$, as desired.

c. Let $\mathcal{U}$ be an orthogonal operator with matrix U. Then $\mathcal{U}$ is unitary $\Leftrightarrow UJ = JU \Leftrightarrow J = U^{-1}JU = U^tJU \Leftrightarrow \mathcal{U}$ is symplectic.

d. L is Lagrangian is equivalent to $0 = \omega(u, v) = -u^tJv$ for all $u, v \in L \Leftrightarrow u$ is orthogonal to Jv for all $u, v \in L \Leftrightarrow J(L) = L^\perp$. The last part follows since J is one to one, and L is dimension n so $J(L)$ is a dimension n subspace orthogonal to L.

3. Consider the linear operator $L = \frac{1}{2}(\star + I) : \Lambda^2((\mathbb{R}^4)^*) \to \Lambda^2((\mathbb{R}^4)^*)$; the two-form β is self-dual if and only if $L\beta = \beta$.

Since $\star^2 = I$ is the identity, $L^2 = \frac{1}{4}(\star^2 + 2\star + I) = \frac{1}{2}(\star + I) = L$ so L is a projection, that is,

$$\{v : Lv = v\} = \text{im}(L).$$

We have

$$L(x_1 \wedge x_2) = \frac{1}{2}(x_1 \wedge x_2 + x_3 \wedge x_4) = L(x_3 \wedge x_4), \quad L(x_1 \wedge x_3) = \frac{1}{2}(x_1 \wedge x_3 - x_2 \wedge x_4) = -L(x_2 \wedge x_4)$$

$$L(x_1 \wedge x_4) = \frac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3) = L(x_2 \wedge x_3),$$

so the space of self-dual forms is 3-dimensional, with basis

$$\{\frac{1}{2}(x_1 \wedge x_2 + x_3 \wedge x_4), \frac{1}{2}(x_1 \wedge x_3 - x_2 \wedge x_4), \frac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3)\}.$$

4. Let $Y = \alpha X + \beta Z$ where $\langle X, Z \rangle = 0$. Then $X \times Y = \beta X \times Z$, since $X \times X = 0$ whereas $\mathcal{D}^{-1}(\star(\mathcal{D}(X) \wedge \mathcal{D}(Y))) = \beta \mathcal{D}^{-1}(\star(\mathcal{D}(X) \wedge \mathcal{D}(Z)))$ since $\mathcal{D}(X) \wedge \mathcal{D}(X) = 0$. It thus suffices to check that

$$X \times Z = \mathcal{D}^{-1}(\star(\mathcal{D}(X) \wedge \mathcal{D}(Z))).$$

We may assume that $X, Z \neq 0$, and by dividing the above by $||X||||Z||$, we may further assume that X and Z are orthonormal. Let X, Z, W be an orthonormal basis for $\mathbb{R}^3$, where $W = X \times Z$, so that the basis defines the standard orientation for $\mathbb{R}^3$. Note that $\mathcal{D}(X), \mathcal{D}(Z), \mathcal{D}(W)$ forms the dual basis. By Lemma 4.3 in the notes, we have that $\star(\mathcal{D}(X) \wedge \mathcal{D}(Z)) = \mathcal{D}(W)$, and so $\mathcal{D}^{-1}(\star(\mathcal{D}(X) \wedge \mathcal{D}(Z))) = W = X \times Z$ as desired.

5. Let $\dim V = n$ and, if $k < n$, let us complete $\{l_1, \ldots, l_k\}$ to a basis $\{l_1, \ldots, l_k, l_{k+1}, \ldots, l_n\}$ of V^*. Now, you have seen in class that the set $\{l_p \wedge l_q,\ 1 \leq p < q \leq n\}$ is a basis for the linear space $\bigwedge^2 (V^*)$ and so the strategy will be to express the sum $l_1 \wedge \lambda_1 + \ldots + l_k \wedge \lambda_k$ purely in terms of this basis. Indeed if we set $\lambda_i = \sum_{j_i=1}^n a_{ij_i} l_{j_i}$ we get, simply by multilinearity

$$l_1 \wedge \lambda_1 + \ldots + l_k \wedge \lambda_k = \sum a_{1j_1} l_1 \wedge l_{j_1} + \ldots \sum a_{kj_k} l_k \wedge l_{j_k}$$

which can be conveniently be written just in terms of two indices $1 \leq i, j \leq n$:

$$= \sum_{i=1}^{k} \sum_{j=1}^{n} a_{ij} l_i \wedge l_j$$

and hence, splitting the range of the index j

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} l_i \wedge l_j + \sum_{i=1}^{k} \sum_{j=k+1}^{n} a_{ij} l_i \wedge l_j.$$

Now, since $\wedge$ is skew-symmetric, it is convenient to rewrite the first of the two sums just in terms of $l_i \wedge l_j$ for $i < j$ as follows

$$l_1 \wedge \lambda_1 + \ldots + l_k \wedge \lambda_k = \sum_{1 \leq i < j \leq k} (a_{ij} - a_{ji})\, l_i \wedge l_j + \sum_{i=1}^{k} \sum_{j=k+1}^{n} a_{ij} l_i \wedge l_j.$$

Therefore $l_1 \wedge \lambda_1 + \ldots + l_k \wedge \lambda_k = 0$ if and only if all the coefficients of the linear combination (of linearly independent vectors) on the right-hand side vanish, namely if $a_{ij} = a_{ji}$ for $1 \leq i < j \leq k$ and $a_{ij} = 0$ for $1 \leq i \leq k$ and $k + 1 \leq j \leq n$. This precisely happens if and only if $\lambda_1, \ldots, \lambda_k \in \operatorname{span}(\lambda_1, \ldots, l_k)$ and the matrix $(a_{ij})_{1 \leq i,j \leq k}$ is symmetric, which is what we had to prove.