Math 52H Homework 7 Solutions

February 29, 2012

1. See forthcoming course notes

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3. The equation $(r-2)^2+z^2=1$ describes a torus in $\mathbb{R}^3$ of small radius 1 centered around a circle of radius 2 in the x-y plane. $T \subset \mathbb{R}^4$ can be imagined as the direct product of two unit circles in two orthogonally intersecting copies of $\mathbb{R}^2$ in $\mathbb{R}^4$ given by (x_1, x_2) and (x_3, x_4), respectively. The first pair (x_1, x_2) should describe which ϕ-slice of the torus we are on, and the second pair (x_3, x_4) should describe where on that ϕ-slice we are. With this geometric intuition in mind, we define the diffeomorphism:

$$\begin{aligned} x_1 &= \cos\phi \\ x_2 &= \sin\phi \\ x_3 &= r-2 \\ x_4 &= z. \end{aligned}$$

First, observe that this is clearly a C^∞ map. So it suffices to check that it is a bijection.

Surjective: As ϕ varies, clearly we hit all (x_1, x_2) on the unit circle. As the equations $x_3 = r-2$ and $x_4 = z$ are linear, and the constraint equations $x_3^2+x_4^2 = 1$ and $(r-2)^2+z^2 = 1$ are identical, we conclude that we hit all possible admissible choices of x_3 and x_4 as well, as r and z vary. ϕ is independent of r, z, so in fact, the map is surjective.

Injective: We must restrict ourselves to $\phi \in [-\pi, \pi)$ to make the map to (x_1, x_2) injective. We don't lose surjectivity in doing this. Then by linearity and the picture we have in our head of $[-\pi, \pi)$ mapping to a unit circle, we see right away that this map is injective.

4. Change to polar coordinates. We have to chop up the domain into that part on which the argument of the absolute value is positive and that part on which it is negative.

$$\frac{x+y}{2} - x^2 - y^2 = r\left(\frac{\cos\theta + \sin\theta}{2} - r\right)$$

so the zero locus of the integrand is when $r = 0$ and when $r = (\cos\theta + \sin\theta)/2$. Thus,

$$\begin{aligned}
\iint_D \left|\frac{x+y}{2} - x^2 - y^2\right| dx, dy &= \int_0^{2\pi}\int_{(\sin\theta+\cos\theta)/2}^{1} r^2\left(r - \frac{\cos\theta+\sin\theta}{2}\right) dr\, d\theta \\
&\quad + \int_0^{2\pi}\int_0^{(\sin\theta+\cos\theta)/2} r^2\left(\frac{\cos\theta+\sin\theta}{2} - r\right) dr\, d\theta \\
&= \int_0^{2\pi} \frac{1}{4} - \frac{1}{3}\frac{\cos\theta+\sin\theta}{2} + \frac{1}{6}\left(\frac{\cos\theta+\sin\theta}{2}\right)^4 d\theta \\
&= \frac{17\pi}{32}
\end{aligned}$$

5. Let R be the region in $\mathbb{R}^2$ described by

$$R = \{x \geq 0, y \geq 0, 1 \leq xy \leq 2, x \leq y \leq 2x\}.$$

Then

$$\mathrm{Vol}(U) = \iint_R (x+y)\, dx\, dy.$$

It is easier to imagine R and make up a nice change of variables than it is to do so with U. The defining equations suggest

$$\begin{aligned}
r &= xy \\
\theta &= y/x,
\end{aligned}$$

but actually to apply the change of variables theorem, we need the map going in the other direction, so invert the map:

$$\begin{aligned}
x &= \sqrt{\frac{r}{\theta}} \\
y &= \sqrt{r\theta}
\end{aligned}$$

Then R pulls back to the region $[1,2] \times [1,2]$ in the (r,θ)-plane. The Jacobian is

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{r\theta}} & \frac{-\sqrt{r}}{2\theta^{3/2}} \\ \frac{\sqrt{\theta}}{2\sqrt{r}} & \frac{\sqrt{r}}{2\sqrt{\theta}} \end{pmatrix}.$$

So, the absolute value of the determinant of the Jacobian comes out to be $\frac{1}{2\theta}$. Finally, we compute

$$\begin{aligned} \mathrm{Vol}(U) &= \iint_R (x+y)\,dx\,dy \\ &= \int_1^2 \int_1^2 \left(\frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}} \right) \frac{\sqrt{r}}{2}\,dr\,d\theta \\ &= \frac{2\sqrt{2}-1}{3} \int_1^2 \frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}\,d\theta \\ &= \frac{8-2\sqrt{2}}{3}. \end{aligned}$$